E. Cartan's Geometric Theory of Partial Differential Equations

ROBERT HERMANN*
Department of Mathematics, Northwestern University, Evanston, Illinois

1. Introduction

Cartan's theory was developed to deal in a coordinate-free, geometric way with questions of existence and uniqueness of local, real-analytic solutions of the systems of partial differential equations arising in differential geometry. It may be regarded as a synthesis and summary of the nineteenth century work on the geometric theory of partial differential equations, associated with such names as Monge, Pfaff, Jacobi, Frobenius, Lie, and Darboux. Many of the intricate and fascinating details of this work are unknown to mathematicians today because of the intervening revision in mathematical thought and concept.

In this article, we mean to present the high points of Cartan's theory in terms of modern differentiable manifold theory, tempered by a personal belief that the extremes of fiber bundle and algebraic notations should be avoided unless they have a geometric meaning. We shall try to present a

* The preparation of this article was supported in part by National Science Foundation Grant GP-2.

point of view that is complementary to that in the well-known books by Cartan [6] and Kähler [1], going back rather to Cartan's early papers [1–5]. To keep this report within bounds, we will not deal with Cartan's theory of infinite Lie groups and prolongations. This aspect of Cartan's work has received more attention lately (Kuranishi [1, 2]), and we feel that someone new to this subject might profitably first study extensively the theory of exterior differential systems, which constitute its foundation.

In line with current practice in differential geometry, we shall state the main ideas in global, C^∞ language, although the main theorems are restricted to local, real-analytic situations because, among other reasons, of a systematic reliance on the Cauchy-Kowalewski existence theorem for partial differential equations. It may be hoped, with the recent upsurge in the theory of partial differential equations, that this situation will be clarified soon. Certainly, one of the difficulties in applying the standard theory of partial differential equations is that only rarely does a geometric problem present a system of partial differential equations of standard type. Now, the principal technical advance inherent in Cartan's theory is that the definition of "singular solution" and "characteristics" of a system of partial differential equations may, in principle, be put on a completely geometric foundation. It is not beyond the limits of plausibility to assume there may ultimately be a unification.

We will concentrate in this report on the geometric, nontechnical aspects of Cartan's work. The theory of partial differential equations (particularly, the nonlinear types encountered in geometry, which are not usually "elliptic" or "hyperbolic") resists formalization, and the reader should be warned that Cartan's theory does not provide an expressway toward even the solutions of the classical, real-analytic, local problems. However, the difficulties here are chiefly in the domain of linear algebra, and this survey has been written in the hope of attracting experts in other fields to the problems. The core of the report is a reformulation in Section 3 of Cartan's main existence theorem that we believe clears up some of the mystery that has surrounded it in the past and that gives a heightened sense of geometric intuition to the subject. The rest of the article treats topics rather randomly, reflecting the incomplete nature of the theory when it left Cartan's hands and as it remains today.

2. Completely Integrable Systems

We first present a summary of the main differential geometric notations that we will use. As basic reference, we use Helgason's book [1], although we will use slightly different notations. For the sake of clear formulation of the ideas, all manifolds, tensor fields, curves, maps, etc., will be of differentiability class C^∞ unless mentioned otherwise. Manifolds will usually be connected and representable as the union of a countable number of compact sets.

Let M be a manifold. For $p \in M$, M_p is the tangent space to M at p. $T(M) = \bigcup_{p\in M} M_p$ is the tangent bundle to M. $C(M)$ denotes the ring of C^∞ real valued functions on M, $V(M)$ the set of vector fields on M. An element $X \in V(M)$ can either be defined as a derivation $f \to X(f)$ of $C(M)$ or as a cross-section map $p \to X(p) \in M_p$ from M to $T(M)$. $V(M)$ is both a $C(M)$-module (since two vector fields can be added and multiplied by functions in an obvious way) and a real (infinite-dimensional) Lie algebra with respect to the Jacobi bracket operation $(X, Y) \to [X, Y]$. If $\varphi : M \to M'$ is a map of manifolds, $\varphi^* : C(M') \to C(M)$ is the dual homomorphism or function:

$$\varphi^*(f')(p) = f'(\varphi(p)) \qquad \text{for} \quad f' \in C(M'), \quad p \in M.$$

For $p \in M$, $\varphi_* : M_p \to M'_{\varphi(p)}$ will be the *differential* of φ at p, defined as follows:

$$M_p = \{v : v \text{ is a linear map } C(M) \to R \text{ such that}$$

$$v(fg) = v(f)g(p) + v(g)f(p)\}.$$

$$\text{For} \quad v \in M_p, \quad \varphi_*(v)(f') = v(\varphi^*(f')) \quad \text{for} \quad f' \in C(M').$$

A *p-differential form*, denoted by ω, is a $C(M)$-multilinear, skew-symmetric map assigning a function $\omega(X_1, ..., X_p)$ for each p-tuple $X_1, ..., X_p$ of vector fields. Recall the various operations connecting vector fields and forms.

(a) *Contraction* (or inner product) of a p-form, ω, by a vector field, X, resulting in a $(p - 1)$-form $X \lrcorner\, \omega$ that is defined as follows:

$$(X \lrcorner\, \omega)(X_1, ..., X_{p-1}) = \omega(X, X_1, ..., X_p).$$

(b) *Exterior product* of a p-form ω_1 and a q-form ω_2, resulting in a $(p+q)$-form $\omega_1 \wedge \omega_2$, satisfying*

$$\omega_1 \wedge \omega_2 = (-1)^{pq}\omega_2 \wedge \omega_1$$

$$X \lrcorner (\omega_1 \wedge \omega_2) = (X_1 \lrcorner \omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge (X \lrcorner \omega_2).$$

If $\theta_i = \sum_{j=1}^m f_{ij}\omega_j$, $1 \leqslant i, j \leqslant m$, where (f_{ij}) is a matrix of functions, $\theta_1, \ldots, \theta_m, \omega_1, \ldots, \omega_m$ are one-forms, then

$$\theta_1 \wedge \ldots \wedge \theta_m = \det(f_{ij})\omega_1 \wedge \ldots \wedge \omega_m.$$

(c) *Exterior derivative*: Assign to each p-form ω a $(p+1)$-form denoted by $d\omega$, satisfying the following rules:

$$d(d\omega) = 0; \qquad d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2, \quad \text{if } \omega_1 \text{ is a } p\text{-form}.$$

For $f \in C(M)$ (we will consider this as a 0-form),

$$df(X) = X(f) \qquad \text{for} \quad X \in V(M).$$

(d) *Lie derivative* of a p-form ω by a vector field X, resulting in a p-form $X(\omega)$, with the following rules:

$$X(d\omega) = dX(\omega); \qquad X(\omega_1 \wedge \omega_2) = X(\omega_1) \wedge \omega_2 + \omega_1 \wedge X(\omega_2);$$

$$X(Y \lrcorner \omega) = [X, Y] \lrcorner \omega + Y \lrcorner X(\omega);$$

$$X(\omega) = X \lrcorner d\omega + d(X \lrcorner \omega);$$

$$X(Y(\omega)) - Y(X(\omega)) = [X, Y](\omega).$$

A (*tangent*) *vector system* on a manifold M is a mapping assigning a subspace of M_p to each $p \in M$, using, say, H_p as the standard notation for this subspace; we shall consider such a system as fixed on M. An *integral submanifold* for the system is defined by a submanifold† $N \subset M$ such that

$$N_p \subset H_p \qquad \text{for each} \quad p \in N.$$

* In writing these relations, we omit explicit mention of the obvious linearity relations. We also omit explicit definition of the exterior product, mentioning in passing that it can now easily be developed using the rules of operation rather than by the usual tensor algebra techniques.

† Technically, a submanifold of M is defined by a map $\varphi : N \to M$ of a manifold N into M which is one-one and whose defferential φ_* is one-one. If φ_* is only

If, in addition, $N_p = H_p$ for all $p \in N$, then we will say that N is a *locally maximal* integral submanifold of the vector field system. A point p is a *nonsingular* point of the system if $\dim H_q$ is constant for q in a neighborhood of p. To avoid extreme sorts of pathology, we will assume that the set of nonsingular points is dense in M and that $\dim H_p$ is constant when p varies over the nonsingular points. A submanifold $T \subset M$ is said to be a *cross section* to the vector system $p \to H_p$ if:

(a) $T_p \cap H_p = (0)$ for all $p \in T$.

(b) $M_p = T_p \oplus H_p$ for all $p \in T$ that are nonsingular for the vector system.

Definition 2.1. A vector system $p \to H_p$ on a manifold M is said to be *completely integrable* at a point $p_0 \in M$ when it has a submanifold of the form $T \times N$ defined in a neighborhood of p_0 such that:

(a) $p_0 = (t_0, n_0)$ for a fixed pair of points $t_0 \in T$, $n_0 \in N$.

(b) $\dim T + \dim N = \dim M$.

(c) For each $t \in T$, such that (t, n_0) is a nonsingular point of the system, the submanifold (t, N) is a locally maximal integral submanifold.

Definition 2.2. A map $\pi : U \to B$ defined on an open set U of M is a *decomposition map* for the vector system $p \to H_p$ if, for each nonsingular point $p \in U$, each $q \in U$ with $\varphi(p) = \varphi(q)$, q is also a nonsingular point, and $\varphi^{-1}(\varphi(p))$ is a locally maximal integral submanifold of the vector system.

There are two standard ways of generating vector systems. First, we can start off with a real subspace $H \subset V(M)$, and let

$$H_p = \{X(p) : X \in H\} \qquad \text{for each} \quad p \in M.$$

Second, we can start off with a linear space $\tilde{H}$ of 1-forms and define H_p to be the annihilator of $\tilde{H}$ at p:

$$H_p = \{v \in M_p : \omega(v) = 0 \qquad \text{for all} \ \omega \in \tilde{H}\}.$$

---

one-one when restricted to each tangent space to N, we speak of an immersed submanifold; the distinction will not be important in this article, and we leave it to the reader to sort out when assertions made about submanifolds can be weakened to immersed submanifolds (roughly, whenever it is really "local" in nature). When no confusion is likely, we will relax the notation to regard N as a subset of M and N_p as a subspace of $M_{\varphi(p)}$.

Systems generated in the first way will be called *vector field systems*, the second type *Pfaffian systems*. This distinction between the dual approaches is necessary if one wants to make geometric sense of the bewildering variety of geometric possibilities for the singularities. In addition, we will see that many concepts are very natural when expressed in one approach, but awkward when viewed dually. In the classical literature, Lie is the champion (and originator) of the vector field approach, whereas Cartan strongly favored differential forms and indeed expressed himself very awkwardly when he was forced to use vectors. Of course, one strong argument for the superiority of forms is that they behave in a much simpler way under mappings. For example, a submanifold N of M is an integral submanifold of a Pfaffian system $\tilde{H}$ on M if each $\omega \in \tilde{H}$ is identically zero when restricted to the submanifold N.

Example A. *Nonsingular completely integrable systems.* We can start off with a vector field system $H \subset V(M)$ such that every point is nonsingular. In addition, we may as well suppose that H is a $C(M)$-submodule of $V(M)$, for, if not, it can be replaced by the $C(M)$-submodule it generates. The Pfaffian approach is just as good, since if $\tilde{H}$ is defined as

$$\{\omega : \omega \text{ is a 1-form, and } \omega(X) = 0 \text{ for all } X \in H\},$$

then $\tilde{H}$ generates the same vector system as does H. It is readily calculated that, if the system $p \to H_p$ is completely integrable, then $[H, H] \subset H$. (An alternate characterization in terms of $\tilde{H}$ is:

For each $\omega \in \tilde{H}$, $d\omega$ lies in the ideal* generated by $\tilde{H}$).

The converse is a basic fact in differential geometry, the *Frobenius complete integrability theorem*. The standard modern reference for this and other basic facts about nonsingular completely integrable systems is in Chevalley's book [1], but, since this was the pioneering effort, it is difficult for the nonspecialist reader to organize the facts contained there. Palais has carried Chevalley's work further and his thesis [1] is the best source for information about completely integrable systems. Many of the basic facts and geometric ideas concerning nonsingular completely integrable systems are due to Ehresmann and his students, particularly Reeb [1] and Haefliger [1]. Although this is not our main

* The exterior product $\wedge$ makes the set of all forms into an algebra, the *Grassmann* algebra on the forms.

concern, we will briefly review some of these facts, with no attempt to assess the history correctly.

Suppose we start off with a nonsingular vector field system $H \subset V(M)$ with $[H, H] \subset H$, such that $\dim H_p = m$ for all $p \in M$. A function $f \in C(M)$ is an *integral* for H if $X(f) = 0$ for all $X \in H$. A coordinate system $(x_1, \ldots x_n)$ for a neighborhood U of M is a *flat* coordinate system for the vector field system if the last $(n - m)$ functions $x_{m+1} \ldots x_n$ of the coordinate system are integrals of H. It is easily seen that then the vector fields $\partial/\partial x_1, \ldots, \partial/\partial x_m$ form a *basis* for H, and that the submanifolds, $x_{m+1}, \ldots, x_n =$ constants, are locally maximal integral submanifolds for H. Definition 2.1 when translated into local coordinates requires precisely that every point of M lie in a neighborhood with such a flat coordinate system.

A proof of the existence of flat coordinate systems can be obtained along the following lines: Choose an arbitrary basis $X_1, \ldots, X_m$ of H (as a $C(M)$-module) valid in a neighborhood of p, and an arbitrary coordinate system about p, $x_1, \ldots, x_n$. We can arrange (at most changing the basis and reordering the coordinate system) that

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} + A_{1,n+1} \frac{\partial}{\partial x_{m+1}} + \ldots + A_{1,n} \frac{\partial}{\partial x_n} \\ &\vdots \\ X_m &= \frac{\partial}{\partial x_m} + A_{m,m+1} \frac{\partial}{\partial x_{m+1}} + \ldots + A_{m,n} \frac{\partial}{\partial x_n} . \end{aligned}$$

The condition $[H, H] \subset H$ now forces:

$$[X_i, X_j] = 0 \qquad \text{for} \quad 1 \leqslant i, j \leqslant m.$$

Now, we can further suppose the coordinate system changed so that

$$X_1 = \frac{\partial}{\partial x_1} .$$

(This requires solving ordinary differential equations, i.e., finding the integral curves of X_1.) Then,

$$[X_1, X_i] = 0 \qquad \text{forces:}$$

$$\frac{\partial A_{i,j}}{\partial x_1} = 0 \qquad \text{for all} \quad 1 \leqslant i \leqslant m, \quad m + 1 \leqslant i \leqslant n.$$

Thus, the vector fields $X_2, \ldots, X_m$ generate a vector field system on the space of $(n - 1)$ variables $x_2, \ldots, x_n$. We can use an induction hypothesis to find a flat coordinate system for this system, which, put together with x_1, gives a flat coordinate system for H.

Having proved the existence of local flat coordinate systems, we can outline the development of the simple general global facts that play a vital role in the foundations of differential geometry. For $p \in M$, let H^p be the set of points of M that can be joined to p by continuous, piecewise C^∞ integral curves of H. The flat coordinate systems can be used to give H^p a submanifold structure, and it is a maximal connected integral manifold of H, called the *leaf* of H passing through p. The system of leaves defines an equivalence relation on M: Two points are equivalent if they lie on the same leaf. The quotient of M by this equivalence relation with the quotient topology is called the *leaf space* or *decomposition space* of the system, denoted by $H \setminus M$. One basic theorem asserts that the projection map $M \to H \setminus M$ is an open map. Palais calls the system *regular* if the leaf space $H \setminus M$ can be made into a (possibly non-Hausdorff) manifold such that the projection $\pi : M \to H \setminus M$ is a map of maximal rank. It is rather important for geometric applications to have available broad sufficient conditions for this to be true. This is a field in which further research is needed; the following theorem can be found in a slightly different form in Hermann [1], and is a beginning towards the solution of this problem.

Suppose that M is a complete Riemannian manifold; that all the leaves of H are closed as subsets of M and are geodesically parallel; and that the holonomy group (in the sense defined in [1]) *of each leaf is trivial. Then, the leaf space of H \ M has a natural manifold structure which is Hausdorff.*

Finally, each leaf H^p has the following geometric property: If $\varphi : N \to M$ is any map of a manifold N into M such that $\varphi(N) \subset H^p$, then it can be factored through a map $N \to H^p$. This fact plays a crucial role in proving that a subalgebra of a finite-dimensional Lie algebra generates a connected subgroup of the corresponding Lie group. Further, the leaves of completely integrable vector field systems that do have singularities do not necessarily have this property, so here is one obstacle to extending even the meager global information that we do have about nonsingular systems to singular ones.

Example B. *Transformation groups.* Consider the following data. A connected Lie group G acts on a manifold M; hence, its Lie algebra $\mathbf{G}$ acts as a Lie algebra of vector fields on M. In this case, the orbits Gp are maximal connected integral submanifolds of the vector field system on M defined by $\mathbf{G}$. The nonsingular points are the points that lie on ***maximal orbits*** of G, i.e., the points where the isotropy subgroups of G have minimal dimension. This situation should be regarded as a "model" for more general sorts of integrable vector field systems. In this direction, Palais [1] has investigated to what extent a finite-dimensional Lie algebra $\mathbf{G}$ of vector fields on a manifold gives rise to a global action of the corresponding simply connected Lie group on G. His main result (and it is a very important, but little-known one) is that this is so if there is a finite set $X_1, ..., X_r$ of vector fields in $\mathbf{G}$ which generate $\mathbf{G}$ as a Lie algebra and such that each of the integral curves of the $X_1, ..., X_r$ can be indefinitely extended.

It is of great importance to the general program to have some information about the orbits and orbit space in this transformation group case. Presumably, information is available in the algebraic geometry literature concerning the case where G is an algebraic group acting on an algebraic manifold, although it is usually not presented in a form that is very useful for geometric purposes. More detailed geometric information can be obtained in case G is a closed group of isometries of a Riemannian metric on M (Hermann [2]), although even here only a beginning has been made. For this case, it is interesting to note that the integral manifolds (i.e., the orbits) have the property that they are of constant distance apart with respect to the Riemannian metric. Reinhart [1] and Hermann [1] have investigated nonsingular systems whose leaves have this property. Finally, some steps have been made toward proving the existence of maximal connected integral manifolds from integrable vector field systems having a "locally finitely generated" property (Hermann [1]).

Example C. *Pfaffian systems.* The problem of treating the singularities of vector systems is so broad that it includes the already enormously difficult problem of the singularities of mappings. To see this, let $\varphi : M \to M'$ be a map of manifolds, and let $\tilde{H}$ be the module of 1-forms generated by all those of the form $\varphi^*(\omega')$, where ω' is a 1-form M'. (A generalization might be to choose ω' from a given Pfaffian system on M'.) Thus, for $p \in M$, $H_p = \{v \in M_p : \varphi_*(v) = 0\}$, hence the nonsingular points of the vector system are the nonsingular points of the mapping.

Given a module $\tilde{H}$ of 1-forms, we can, of course, form the dual vector field system:

$$H = \{X \in V(M) : \omega(X) = 0 \qquad \text{for all } \omega \in \tilde{H}\},$$

but, of course, there is no reason why integral manifolds of $\tilde{H}$ should be integral manifolds of H.

If this is so, we have available, at least in favorable cases, the techniques outlined in Example B. There does not seem to be any general conditions for this to be so known, although some work in a special case has been done in Hermann [1]. (We may also remark that this question is related to the proper definition of the idea of a *nondegenerate* singularity for a Pfaffian system. Take, for example, the simplest case where $\tilde{H}$ is generated by df, for some $f \in C(M)$. A singular point would be a point where $df(p) = 0$, i.e., f would have a critical point at p. f would have a nondegenerate singularity at p in Morse's sense if a coordinate system $(x_1, ..., x_n)$ valid in the neighborhood of p could be chosen so that

$$f = f(p) + \sum_{i=1}^{n} a_i x_i^2, \qquad \text{with each} \quad a_i \neq 0.$$

Then,

$$df = \sum_{i=1}^{n} 2a_i x_i \, dx_i.$$

A vector field $X = \sum_{i=1}^{n} A_i \, \partial/\partial x_i$ satisfies: $df(X) = 0$ if $\sum_{i=1}^{n} a_i x_i A_i = 0$. Consider the Taylor expansion of the functions A_i:

$$A_i = A_i(p) + \sum_{j=1}^{n} A_{i,j}(p) x_j + \dots .$$

Thus, $a_i A_i(p) = 0$; hence, $A_i(p) = 0$ for $i = 1, ..., n$. Thus, the point p is also an isolated singular point for H also; hence, the "duality" works in this case.)

Pursuing these points further here would take us too far afield. In closing this section, we can only refer to the work of Reeb [1] and Haefliger [1] on completely integrable Pfaffian systems generated by a single one-form; this work could serve as a model for extension to more complicated systems.

3. The Fundamental Concepts of Cartan's Theory

A manifold M will be fixed throughout this section. Let I be a differential ideal of differential forms on M. Thus, I is an ideal in the sense of the exterior algebra of forms, and $dI \subset I$. A submanifold $N \subset M$ is an *integral submanifold* of I if each $\omega \in I$ is identically zero when restricted to N, i.e.,

$$\varphi^*(\omega) = 0,$$

where φ is the inclusion map of N into M. Let us form the *linear variational equations* of the underlying differential equations defining N as follows.

Let φ_t be a one-parameter family of maps of N into M. We can form a vector field $\mathbf{v}$ on N pointing out into N (i.e., a map $\mathbf{v} : N \to T(M)$ such that $\mathbf{v}(p) \in M_{\varphi(p)}$ for $p \in N$) in the following way: For $p \in N$, $\mathbf{v}(p)$ is the tangent vector to the curve $t \to \varphi_t(p)$ at $t = 0$. $\mathbf{v}$ is called the *infinitesimal deformation of* N. If each map φ_t is an integral submanifold of N, the vector field $\mathbf{v}$ will satisfy a system of *linear* differential equations.

THEOREM 3.1. *Suppose φ_t is a one-parameter family of integral manifolds of the differential ideal I. Then its infinitesimal deformation* $\mathbf{v}$ *satisfies*

$$d(\varphi_0^*(\mathbf{v} \lrcorner \omega)) + \varphi_0^*(\mathbf{v} \lrcorner d\omega) = 0 \tag{3.1}$$

for all $\omega \in I$.

Some explanation of this formula is necessary. For $p \in N$, $\mathbf{v}(p) \in M_{\varphi(p)}$ and $\omega(\varphi(p))$ is a covector at p, hence, $\mathbf{v}(p) \lrcorner \omega(\varphi(p))$ is the $(p - 1)$-covector resulting from contracting ω by $v(p)$; $(\mathbf{v} \lrcorner \omega)(p)$ is then this covector, i.e.,

$$(\mathbf{v} \lrcorner \omega(p))(v_1, \ldots, v_{p-1}) = \omega(\mathbf{v}(p), v_1, \ldots, v_{p-1})$$

for $v_1, \ldots, v_{p-1} \in M_{\varphi(p)}$. Restricting this covector field to N via the map φ_0 defines the differential form $\varphi_0{}^*(\mathbf{v} \lrcorner \omega)$ on N.

To prove (3.1), we will actually prove the following more general formula, valid for any form ω and any one-parameter family of maps:

$$\frac{\partial}{\partial t} \varphi_t^*(\omega)|_{t=0} = d(\varphi_0^*(\mathbf{v} \lrcorner \omega)) + \varphi_0^*(\mathbf{v} \lrcorner d\omega). \tag{3.2}$$

Notice now that:

(a) It suffices to prove the formula locally.

(b) If ω is a 0-form, i.e., a function, then the formula is true, by the very definition of v.

(c) If the formula is true for ω, it is true for $d\omega$. To see this, take the exterior derivative of both sides of the formula. The left-hand side is $(\partial/\partial t)(\varphi_t{}^*(d\omega))$, while the right-hand side is $d(\varphi^*(\mathbf{v} \lrcorner d\omega))$, which is what it should be to prove (c).

(d) If it is true for ω_1 and ω_2, it is true for $\omega_1 \wedge \omega_2$ and $\omega_1 + \omega_2$. Only this remark requires some calculation, using the standard formula:

$$\mathbf{v} \lrcorner (\omega_1 \wedge \omega_2) = (\mathbf{v} \lrcorner \omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge (\mathbf{v} \lrcorner \omega_2),$$

if ω_1 is a p-form (and the similar formula for $d(\omega_1 \wedge \omega_2)$).

$$\begin{aligned} d\varphi^*(\mathbf{v} \lrcorner (\omega_1 \wedge \omega_2)) &= d\varphi^*((\mathbf{v} \lrcorner \omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge (\mathbf{v} \lrcorner \omega_2) \\ &= d\varphi^*(\mathbf{v} \lrcorner \omega_1) \wedge \varphi^*(\omega_2) + (-1)^{p-1} \varphi^*(\mathbf{v} \lrcorner \omega_1) \wedge \varphi^* d\omega_2 \\ &\quad + (-1)^p \{ d\varphi^*(\omega_1 \wedge \varphi^*(\mathbf{v} \lrcorner \omega_2) \\ &\quad + (-1)^p \varphi^*(\omega_1) \wedge d\varphi^*(\mathbf{v} \lrcorner \omega_2) \}. \end{aligned}$$

$$\begin{aligned} \varphi^*(\mathbf{v} \lrcorner d(\omega_1 \wedge \omega_2)) &= \varphi^*(\mathbf{v} \lrcorner d\omega_1) \wedge \varphi^*(\omega_2) + (-1)^{p+1} \varphi^*(d\omega_1 \wedge (\mathbf{v} \lrcorner \omega_2)) \\ &\quad + (-1)^p \varphi^*((\mathbf{v} \lrcorner \omega_1) \wedge d\omega_2 + (-1)^p (\omega_1 \wedge (\mathbf{v} \lrcorner d\omega_2))). \end{aligned}$$

Adding $d\varphi^*(v \lrcorner (\omega_1 \wedge \omega_2))$ and $\varphi^*(v \lrcorner d(\omega_1 \wedge \omega_2))$, we see that the terms that should cancel to prove the first part of (d) do, i.e., we get

$$(d\varphi^*(\mathbf{v} \lrcorner \omega_1) + \varphi^*(\mathbf{v} \lrcorner d\omega_1)) \wedge \varphi^*(\omega_2) + \varphi^*(\omega_1) \wedge (d\varphi^*(\mathbf{v} \lrcorner \omega_2) + \varphi^*(\mathbf{v} \lrcorner d\omega_2)).$$

The formula for $\omega_1 + \omega_2$ is obvious. Theorem 3.1 is now proved, since locally any differential form can be built up from functions by using d, $\wedge$, and $+$.

Let $G^m(M)$, $m \geqslant 0$, denote the Grassmann bundle of p-dimensional subspaces of the tangent bundle of M. $G^0(M)$ will be identified with M. If U is an open subset of $G^m(M)$, an *m-vector field*, denoted usually by $\mathbf{X}$, is a map $U \to T(M)$ that commutes with the projection of U and $T(M)$ on M. Thus, for each p in M that lies in the projection of U, $\mathbf{X}$ assigns a tangent vector $\mathbf{X}(\gamma) \in M_p$ to each p-dimensional subspace $\gamma \in M_p$ that lies in U. Let $V(U, M)$ denote the set of these objects.

Suppose $\dim N = m < \dim M$, and let $\varphi : N \to M$ be a submanifold map. For each $p \in N$, $\varphi_*(N_p)$ is an m-dimensional subspace of $M_{\varphi(p)}$, hence defines an element of $G^m(M)$. If no confusion is likely, we will use φ_* also to denote this "Gauss map" of $N \to G^m(M)$. Given an m-vector field $\mathbf{X}$, a one-parameter family $t \to \varphi_t$ of submanifold maps of N into M is an *integral deformation* for $\mathbf{X}$ if:

For each $p \in M$, $\varphi_{t^*}(p)$, as an element of $G^m(M)$, belongs to the domain of $\mathbf{X}$, and

$$\mathbf{X}(\varphi_{t^*}(p)) = \frac{\partial}{\partial t}\varphi_t(p) \quad (= \text{the tangent vector to the curve } u \to \varphi_u(p) \text{ at } u = t).$$

THEOREM 3.2. *Suppose that U is an open subset of $G^m(M)$, $\mathbf{X} \in V(U, M)$, and $\varphi : N \to M$, is a submanifold such that $\varphi_*(N) \subset U$. Suppose in addition that all the data are real analytic. Then, there is, locally, a unique deformation φ_t that is an integral deformation of X and such that $\varphi_0 = \varphi$. φ_t is defined locally by a Cauchy-Kowalewski system of partial differential equations solved for the first partial derivatives with respect to t. The initial conditions for $t = 0$ are determined by φ.*

Proof. Suppose $\dim M = n$, $\dim N = m$. Adopt the following ranges of indices and summation conventions: $1 \leqslant i, j, ..., \leqslant n$; $1 \leqslant a, b, ..., \leqslant m$; $m + 1 \leqslant u, v, ..., \leqslant n$. Since we are only working locally, suppose that (y_i) (resp. (x_a)) is a coordinate system for M (resp. N). We can suppose that φ_0, the initial map of N into M, is defined by

$$\varphi(x_1, ..., x_p) = (y_i(x_a)).$$

Then,

$$\varphi_*\left(\frac{\partial}{\partial x_a}\right) = \frac{\partial y_j}{\partial x_a}\frac{\partial}{\partial y_j}.*$$

If U is a sufficiently small neighborhood of $\varphi_*(N)$ in $G^m(M)$, a mapping from an open set of the space of variables (y, v_{0j}), with rank $(v_{ij}) = p$, onto U can be defined by assigning the p-dimensional subspace γ of M_y spanned by $v_{ij}(\partial/\partial y_j)$. An $\mathbf{X} \in V(M, V)$ then defines real-valued functions $X_k(y, v_{ij})$ of the indicated variables so that

$$\mathbf{X}(\gamma) = X_k(y, v_{ij})\frac{\partial}{\partial y_k}.$$

We can look for φ_t on the form: $\varphi_t(x_1, ..., x_p) = (y_i(x, t))$. Thus, φ_t satisfies $\partial\varphi_t/\partial t = \mathbf{X}(\varphi_{t^*})$, i.e., φ_t is an integral deformation of $\mathbf{X}$, if and only if

$$\frac{\partial y_k(x, t)}{\partial t} = X_k\left(y(x, t), \frac{\partial y_i(x, t)}{\partial x_a}\right).$$

* $\partial/\partial y_i$ (resp. $\partial/\partial x_a$) are the vector fields on M (resp. N) that form a dual basis to the 1-forms dy_i (resp. dx_a), i.e., that satisfy $dy_i(\partial/\partial y_j) = \delta_{ij}$ (resp. $dx_a(\partial/\partial x_b) = \delta_{ab}$).

Since this is a Cauchy-Kowalewski system, the proof of Theorem 3.2 is complete.

Let I be a differential ideal of differential forms on a manifold M. A subspace $\gamma \subset M_p$, for $p \in M$, is said to be an *integral element of* I if all the $\omega \in I$ are identically zero when restricted to γ. Let $G^m(I)$ be the set of all m-dimensional integral elements. The necessary and sufficient condition that a submanifold be an integral submanifold of the ideal I is that all its tangent spaces be integral elements of I. If all the data are real analytic and if I is locally finitely generated (which, at the present time, are the only general hypotheses which give nontrivial results), $G^m(I)$ is a real-analytic subset of $G^m(M)$, i.e., is defined locally by setting a finite number of real-analytic, real-valued functions on $G^m(M)$ equal to zero, and there are then obviously points of $G^m(I)$ about which $G^m(I)$ is a submanifold of $G^m(M)$.

If $\gamma \subset M_p$ is an integral element of I, a $v \in M_p$ is said to be *in involution* with γ if the subspace of M_p spanned by v and γ is an integral element of I. In other words, *$v \lrcorner \omega$ is identically zero when restricted to γ, for each $\omega \in I$.*

The set of $v \in M_p$ that are in involution with a fixed integral element $\gamma \subset M_p$ is a linear subspace of M_p. Call it $i(\gamma)$. Clearly, $\gamma \subset i(\gamma)$. Let $r(\gamma) = \dim i(\gamma) - \dim \gamma - 1$. Then, $r(\gamma)$ geometrically is the dimension of the set of integral elements of I of one-higher dimension containing γ (with the convention that the dimension of the empty set is -1).

Let U be an open subset of $G^m(M)$. An $\mathbf{X} \in V(U, M)$ is said to be *in involution* with I if

$$\mathbf{X}(\gamma) \text{ is in involution with } \gamma \text{ for each } \gamma \in U \cap G^m(I)$$

THEOREM 3.3. *Suppose that:*

(a) *All the data are real analytic and I is a finitely generated differential ideal of forms on M.*

(b) *$\varphi : N \to M$ defines an integral manifold of I.*

(c) *U is an open set of $G^m(M)$ such that $\varphi_t(N) \subset U$.* $\operatorname{Dim} N = m$.

(d) *$U \cap G^m(I)$ is a submanifold of U, i.e., the system of real-valued functions on G^m whose vanishing defines the elements of $G^m(I)$ have only regular points in $U \cap G^m(I)$.*

(e) *$\mathbf{X} : U \to T(M)$ is a map commuting with the projections on M (i.e., an element of $V(U, M)$) which is in involution with I.*

(f) *φ_t is the integral deformation of $\mathbf{X}$ such that $\varphi_0 = \varphi$.*

Then, for each t, φ_t is an integral manifold of I. The map $(x, t) \to \varphi_t(x)$ of $N \times [0, 1] \to M$ is an integral map of I. It is an integral manifold of dimension $(p + 1)$ containing φ if the following condition is satisfied:

$$\mathbf{X}(\gamma) \notin \gamma \qquad \textit{for each} \quad y \in G^m(I).$$

If the following further condition is satisfied, it is locally the only integral manifold of dimension $(p + 1)$ containing φ:

For each γ in a neighborhood of $\varphi_*(N)$ in $G^m(I)$,

$$r(\gamma) = 0.$$

Proof. The proof of the last two remarks should be obvious from the definitions. The proof of the rest can be found, with a certain amount of difficulty, in the books by Cartan [6] and Kähler [1]. For clarity, we give the proof here only in the case that I is defined by a nonsingular Pfaffian system, i.e., we suppose that I is generated by a set of everywhere independent 1-forms. Most of the difficulties with multilinear algebra are minimized in this case, and the Pfaffian system case is by far the most important in applications of the theory.

Suppose that $\dim M = n$. Adopt the following range of indices and the corresponding summation conventions: $1 \leqslant i, \quad j, \ldots \leqslant n$; $1 \leqslant a, b, \ldots \leqslant r$; $r + 1 \leqslant u, v, \ldots \leqslant n$; $1 \leqslant \alpha, \beta, \ldots \leqslant m$. Since the result is local, we can suppose that (ω_j) is a basis of 1-forms for M and that the (ω_a) generate I as a differential ideal. Let (Y_i) be a dual basis of fields on M, i.e., $\omega_i(Y_j) = \delta_{ij}$, and let (X_a) be a basis of vector fields on N. Determine a mapping of the Cartesian product of M with the space of real variables $(v_{\alpha i})$ (such that the matrix $(v_{\alpha i})$ has rank m) into $G^m(M)$ by mapping $(y, v_{\alpha i})$ into the subspace γ spanned by the vectors $v_{\alpha i} Y_i(y)$. $\mathbf{X}$, as a mapping of $G^m(M) \to T(M)$, is determined by real-valued functions $A_k(y, v_{\alpha i})$ of the indicated variables such that

$$\mathbf{X}(\gamma) = A_k(y, v_{\alpha i}) Y_k(y).$$

By hypothesis, we are working in a small enough open subset of $G^m(M)$ so that the portion of the subset composed of integral elements of I is a submanifold of $G^m(M)$. Now, if γ is the subspace of M_y spanned by $v_{\alpha i} Y_i(y)$, γ is an integral element of I, if and only if

$$v_{\alpha a} = 0, \qquad c_{uva}(y) v_{\alpha u} v_{\beta v} = 0$$

where c_{jki} are the real-valued functions on M such that: $d\omega_i = c_{jki}\omega_j \wedge \omega_k$. The equations $v_{aa} = 0 = c_{uva}v_{\alpha u}v_{\beta v}$ thus define, in the space $(y, v_{\alpha i})$, the inverse image of the set of integral elements of $G^m(I)$, and by hypothesis we are only working in a subset of this space where these equations are regular. Then, the conditions that $\mathbf{X}(\gamma)$ be in involution with γ whenever γ is an integral element are the existence of functions $B_{a\alpha\beta}(y, \mathrm{v})$, $C_{a\alpha\beta b}(y, \mathrm{v})$, $D_{a\alpha\beta b}(y, \mathrm{v})$, $E_{a\alpha\beta\gamma b}(y, \mathrm{v})$ such that the following relations hold identically in y and v*

$$A_a(y, \mathrm{v}) = B_{a\alpha b}(y, \mathrm{v})v_{\alpha b} + C_{a\alpha\beta b}(y, \mathrm{v})C_{uvb}(y)v_{\alpha u}v_{\beta v}.$$

$$C_{jka}(y)A_j(y, \mathrm{v})v_{\alpha k} = D_{a\alpha\beta b}(y, \mathrm{v})v_{\beta b} + E_{a\alpha\beta\gamma b}C_{uvb}(y)v_{\beta u}v_{\gamma v}.$$

Suppose that (θ_α) is a basis for 1-forms of N. Suppose further that $\varphi_t : N \to M$ is a one-parameter family of immersions which is an integral deformation of $\mathbf{X}$, i.e., satisfies $\partial\varphi_t/\partial t = \mathbf{X}(\varphi_{t^*})$. Suppose that

$$\varphi_t^*(\varphi_i) = u_{\alpha i}(x, t)\theta_\alpha.$$

Let $\mathbf{u}(x, t)$ be the matrix $(u_{\alpha i}(x, t))$. Then a short calculation shows that

$$\mathbf{X}(\varphi_{t^*}(x)) = A_k(\varphi_t(x), u_{\alpha i}(x, t))Y_k(\varphi_t(x)) \qquad \text{for} \quad x \in N.$$

By (2.2),

$$\begin{aligned}
\frac{\partial}{\partial t}\varphi_t^*(\omega_i)(x) &= \frac{\partial u_{\alpha i}}{\partial t}\theta_\alpha(x) \\
&= \varphi_{t^*}(\mathbf{X}(\varphi_t^*) \lrcorner\, d\omega_i) + d\varphi_t^*(\mathbf{X}(\varphi_{t^*}) \lrcorner\, \omega_i)(x) \\
&= C_{jki}(\varphi_t(x)A_k(\varphi_t(x), \mathbf{u}(x, t))u_{\alpha j}(x, t)\theta_\beta(x) + d(A_s(\varphi_t(x), \mathbf{u}(x, t)).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\varphi_t^*(d\omega_a) &= \frac{\partial}{\partial t}\varphi_t^*(C_{jka}\omega_j \wedge \omega_k) \\
&= \frac{\partial}{\partial t}[\varphi_t^*(C_{jka})u_{\alpha j}u_{\beta k}]\theta_\alpha \wedge \theta_\beta \\
&= \text{using (2.2) again,} \qquad d\varphi_t(\mathbf{X}(\varphi_t^*) \lrcorner\, d\omega_a) \\
&= d[\varphi_t^*(C_{jka})A_j(\varphi_t(x), \mathbf{u}(t, x))u_{\alpha k}(x, t)\theta_\alpha].
\end{aligned}$$

* v will denote the matrix $(v_{\alpha i})$. Note that in asserting the existence of the functions B, C, D, and E, we are just using the following general fact: If $f_1, \ldots, f_N$ are real-valued functions on a manifold such that the rank of the 1-forms $df_1, \ldots, df_N$ is constant, and such that they are all zero at a certain point, any function g which vanishes at all the zeros of the f's neighborhood of this point can be in this neighborhood, be written in the form: $g = g_1f_1 + \cdots + g_Nf_N$.

Combine these relations and put $(\partial/\partial t)(u_{\alpha k}(x, t))$ and

$$\frac{\partial}{\partial t}(C_{jka}(\varphi_t(x))u_{\alpha j}(x, t(x))u_{\beta k}(x, t))$$

on the left-hand side, everything else on the right-hand side. It is seen that *together* they give a system of linear homogeneous Cauchy-Kowalewski partial differential equations. The initial data at $t = 0$ is zero, since the vanishing of these terms at $t = 0$ expresses the fact that φ_0 is an integral manifold of I. Then, by the uniqueness of such systems, the $u_{\alpha a}(x, t)$, in particular, vanish for all t, i.e., each φ_t is an integral manifold of I. This finishes the proof of Theorem 3.3.

Thus, the key problem is the purely algebraic question of the existence of $\mathbf{X} \in V(U, M)$ satisfying the hypotheses of Theorem 3.3; if there is a plentiful supply of such objects we can build up integral manifolds by induction on the dimension, starting with 0-dimensional integral manifolds, i.e., points. We now turn to the formulation of Cartan's method for dealing with this problem. In turn, this leads to the distinction between the singular and nonsingular integral manifolds.

In addition to the differential ideal I, we will suppose that M carries a fixed field $T : p \to T_p \subset M_p$ of tangent subspaces of constant dimension (which may be zero). A subspace $\gamma \in M_p$ will be said to be *transversal* to T if $\gamma \cap T_p = (0)$. Notice that the set of elements of $G^m(M)$ transversal to T is open in $G^m(T)$. If U is an open subset of $G^m(M)$, an $\mathbf{X} \in V(U, M)$ will be said to be *transversal to* T if $\mathbf{X}(\gamma) \notin T_p$ for each $\gamma \in U, \gamma \in M_p$, that is itself transversal to T. A submanifold $\varphi : N \to M$ is *transversal to* T if $\varphi_*(N_p)$ is transversal to $T_{\varphi(p)}$ for each $p \in N$. Until further notice, when we say "integral element" or "integral submanifold" we mean one that is transversal to T. We will only be considering open sets $U \subset G^m(M)$ composed of subspaces transversal to T. For an integral element $\gamma \subset M_p$, let

$$r(\gamma, T) = \dim(i(\gamma)/i(\gamma) \cap T_p + \gamma) - 1.$$

$r(\gamma, T)$ is then the dimension of the space of integral elements of dimension one greater than γ containing γ that are transversal to T.

An integral element $\gamma_0 \in G^m(I)$ is said to be *regular* if there is a neighborhood U of γ_0 in $G^m(M)$ such that

$$r(\gamma_0, T) = \min\{r(\gamma, T) : \gamma \in G^m(I) \cap U\},$$

and $G^m(I) \cap U$ is a submanifold of U, i.e., the equations defining $G^m(I)$ are nonsingular in U.

THEOREM 3.4. *Suppose that U is a sufficiently small open subset of $G^m(M)$ such that $U \cap G^m(I)$ is a submanifold of $G^m(M)$ which is composed of only regular integral elements transversal to T. In particular, $r(\gamma, T)$, for $\gamma \in G^m(I)$, is constant, equal to, say, r. Then, there are elements $\mathbf{X}_1, \ldots, \mathbf{X}_r \in V(U, M)$ transversal to T such that*

For $\gamma \in U \cap G^m(I)$, $\mathbf{X}_1(\gamma), \ldots, \mathbf{X}_r(\gamma) \in i(\gamma)$, and their images in $i(\gamma)/\gamma$ are linearly independent. Hence, if all the data are real analytic and $r \geqslant 1$, there passes, at least locally, through each integral manifold φ of dimension p whose tangent spaces lie in U an integral manifold of dimension $(p+1)$ which is also transversal to T. In Cartan's language [2], the "Cauchy problem" is solvable for φ. In turn, Cartan's "Cauchy problem" is just the natural geometrization of the standard Cauchy problem for partial differential equations.

Proof. We can suppose that M has a Riemannian metric, i.e., each M_p has a positive definite quadratic form that varies smoothly with p. A smooth vector bundle can be defined on U by assigning to each $\gamma \in G^m(M)$ with $\gamma \subset M_p$ the orthogonal complement of $\gamma + T_p$ in M_p, $(\gamma + T_p)^\perp = \gamma^\perp \cap T_p{}^\perp$. Since $\dim i(\gamma)$ is constant for $\gamma \in U$, this vector bundle splits over $U \cap G^m(I)$ into $i(\gamma)^\perp \cap (\gamma + T_p)^\perp)$ and $i(\gamma) \cap (\gamma + T_p)^\perp$. The dimension of the second component is r. Choose $\mathbf{X}_1, \ldots, \mathbf{X}_r$ as cross sections of this second vector bundle, and extend them over U using the fact that $U \cap G^m(I)$ is a submanifold of U. q.e.d.

The regular integral elements transversal to T are open in $G^m(I)$.

Proof. Assign the vector space $M_p/(\gamma + T_p)$ to each $\gamma \in G^m(I)$ that is a subspace of M_p, $p \in M$. This determines a vector bundle on $G^m(I)$. It is easily seen that there are linear forms on the bundle, varying continuously with γ, having the property that a $v \in M_p/(\gamma + T_p)$ vanishes on all these forms, if and only if $v \in i(\gamma)/(\gamma + T_p \cap i(\gamma))$. Thus, a regular integral element is characterized by the property that a maximal number of these forms is linearly independent, which is then clearly true for a whole neighborhood of γ in $G^m(I)$.

Definition. The exterior differential system defined by the differential ideal I and the tangent subspace field T is said to be of *genus* g if:

(a) Every integral element of dimension less than g that is transversal to T is contained in at least one integral element of dimension g that is transversal to T, and

(b) At least one integral element of dimension g that is transversal to T is not contained in any integral element of higher dimension transversal to T.

Thus, $g \leqslant \dim M - \dim T$. If $g = \dim M - \dim T$, we say that the system is in *involution*. A nested chain of integral elements $\gamma_m \supset \gamma_{m-1} \supset \dots \supset \gamma_1 \supset (0)$, each of one lower dimension, is said to be a *regular chain of integral elements* if each is a regular integral element. If $m \leqslant g$, the genus, it is clear by induction on m that such chains exist. We now have proved the following theorem:

THEOREM 3.5. *If $\gamma_m \supset \gamma_{m-1} \supset \dots \supset (0)$ is a regular chain of integral elements, all tangent to $p \in M$, and if all the data are real analytic, there is a chain of integral submanifolds $\varphi_m \supset \varphi_{m-1} \supset \dots \supset p$, each of one lower dimension, whose tangent spaces at p are the $\gamma_m, \dots, (0)$. In particular, there exist a plentiful supply of integral manifolds of I that are transversal to T.*

It is appropriate at this point to mention the main situation where such a tangent space field T arises in a natural way. Suppose that M is the product of two manifolds B and C, and that T is the tangent-space field composed of the tangent spaces to C, i.e., T is the family of tangent spaces to the fibers of the projection mapping $M \to B$. A submanifold of M of the same dimension as B that is transversal to T is then, at least locally, just a graph of a mapping of B into C. Hence, if I is a differential ideal corresponding to a system of partial differential equations for mappings of B into C, and if the system formed by I and T is *in involution* in the above sense, there is a plentiful supply of mappings of C into B that satisfy the partial differential equations.

Thus, Cartan's theory can make precise the idea that a system of partial differential equations is "in involution," a concept that is not so precisely defined in the classical theory of partial differential equations, or at least seems ill-defined by modern standards.

An integral element $\gamma \in G^m(I)$ is said to be *nonsingular* if it can be imbedded in a regular integral chain whose length is equal to $g + 1$. An integral submanifold of I transversal to T is said to be nonsingular (resp. regular) if all its tangent spaces are nonsingular (resp. regular). It is said to be a singular (resp. nonregular) integral manifold if it is not nonsingular (resp. is not regular). Another of the major accomplishments of Cartan's theory is that the concept of "singular solution" of a system of partial differential equations can be formulated so easily! However, the theory of the singular solutions is still in a very fragmentary form. For example, certain obvious questions seem to be unanswered, and even unexplored: What is the relation between integral elements that are regular with respect to a given T and those that are regular with respect to another T, for example, $T = (0)$? Can a singular integral manifold be regular, i.e., when can a regular integral element be imbedded in a regular integral chain?

Cartan at least partially computed these invariants for many interesting exterior systems, presumably with the aim of formulating general theorems about singular solutions. In particular, he seems to have had foremost in his mind the idea of describing the characteristics of a system of partial differential equations as certain types of singular integral manifolds of the exterior differential system associated with the partial differential equations.

Definition. Let I be a differential ideal of forms and let I' be an ideal containing I such that every integral element of dimension p of I' is nonregular when considered as an integral element of I. Then, the p-dimensional integral manifolds of I' are said to be the *characteristics of I defined by I'*.

Example 1. *A first-order ordinary differential equation.*

M is the space of variables (x, y, y'),

$$\omega = dy - y'\,dx, \qquad d\omega = dx \wedge dy'.$$

f is a given function, $f(x, y, y')$, I is the differential ideal of forms generated by f and ω. T is defined by: $dx = 0$.

The exterior differential system defined by the ideal I is that corresponding to the differential equation $f(x, y(x), dy/dx) = 0$ in the sense that:

(a) If $y(x)$ is a solution of the differential equation, $x \to (x, y(x), y'(x) = dy/dx)$ is an integral manifold of the ideal I.

(b) A one-dimensional integral manifold of I that is transversal to T can be reparameterized so as to arise via (a) from a solution of the differential equation.

A zero-dimensional integral element is a point $(x_0\ y_0, y_0')$ such that $f(x_0, y_0, y_0') = 0$, i.e., a set of initial data for the differential equation. A one-dimensional integral element γ at (x_0, y_0, y_0') transversal to T is defined by a relation of the form: $dy - a\,dx = 0 = dy' - b\,dx$. But,

$\omega = 0$ on γ requires that $a = y_0'$. Now, $df = f_x\, dx + f_y\, dy + f_{y'}\, dy'$*; hence, b must satisfy $f_x + f_y y_0' + f_{y'} b = 0$ at (x_0, y_0, y_0'). b is uniquely determined except when $f_{y'}(x_0, y_0, y_0') = 0$, i.e., (x_0, y_0, y_0') is a regular integral element if $f_{y'}(x_0, y_0, y_0') \neq 0$. However, the differential ideal I' obtained from I by adding the function $f_{y'}$ determines the "characteristics"; we recognize that they are just the envelopes of the "regular" solutions of the differential equations, which are in turn just the solutions that can be obtained by solving for dy/dx.

Notice another fact about this example that is significant for possible generalization: If $f_y \neq 0$, the characteristics can be defined by the ordinary equations $f = f_{y'} = f_x + f_y y' = 0$, i.e., by a zeroth order differential equation. For, if $y(x), y'(x)$ is a curve satisfying these relations, we can prove that $y'(x) = dy/dx$; hence, $y(x)$ is a solution of the differential equation. For,

$$0 = \frac{d}{dx}\left(f(x, y(x), y'(x)\right) = f_x(x, y(x), y'(x)) + f_y(x, y(x), y'(x))\frac{dy}{dx}.$$

This is the germ of the idea behind the well-known fact that the characteristics of a second-order partial differential equation are defined by a first-order partial differential equation.

4. The Cauchy Characteristics

We will continue to work on a manifold M, and I will denote a differential ideal of differential forms on M. Points of M will generally be denoted by p, forms of I by $\omega, \omega_1, \ldots$. In Section 3, we have, roughly, defined an integral element γ as nonregular if there pass an unusually high number of integral elements of higher dimension through this element. We now turn to the *Cauchy characteristics*, which are integral elements that are, in a sense, "maximally" nonregular since a maximal number of integral elements of higher dimension pass through them. Cartan gave this name to them because, in the case of the exterior differential system defining a solution of a first-order partial differential equation, they define the characteristic curves used originally by Cauchy to solve the equation. Further, exterior systems possessing a plentiful supply of Cauchy characteristics are those which it is known can be

* Subscripts mean partial derivatives here.

solved by using ordinary differential equations; in particular, they can be solved in nonanalytic circumstances. However, the characteristics occurring in the theory of second-order partial differential equations are not of this Cauchy type.

A tangent vector $v \in M_p$ is said to be a *Cauchy characteristic* (C-characteristic, abbreviated) vector of the ideal I if

$$v \lrcorner \omega \in I_p \qquad \textit{for all} \quad \omega \in I.$$

(I_p denotes the set of values of the $\omega \in I$ at p.) Similarly, a vector field $X \in V(M)$ is said to be a C-characteristic if

$$X \lrcorner \omega \in I \qquad \textit{for all} \quad \omega \in I.$$

The set of all C-characteristic vectors forms a linear subspace $C(I, p) \subset M_p$. Similarly, the set of all C-characteristic vector fields forms a subspace $C(I)$ of vector fields that is a $C(M)$-submodule of $V(M)$.

If $X \in C(I)$, $X(I) \subset I$.

Proof. For $\omega \in I$, $X(\omega) = d(X \lrcorner \omega) + X \lrcorner d\omega \in I$.

If X, $Y \in C(I)$, $[X, Y] \in C(I)$, i.e., $C(I)$ *is a Lie subalgebra of* $V(M)$.

Proof. For $\omega \in I$, by (3.2), $X(Y \lrcorner \omega) \in I$. But,

$$X(Y \lrcorner \omega) = [X, Y] \lrcorner \omega + Y \lrcorner (X(\omega)).$$

The first and third terms belong to I, hence the second does also.

The characteristic system $p \to C(I, p)$ provides us with a typical example of a vector system on a manifold that satisfies the formal conditions for complete integrability. (It is not known yet under what conditions the actual definition of complete integrability is satisfied at a singular point.) However, we will not bother here with the possible singularities, or even with the complications introduced by the global theory of nonsingular completely integrable systems, but shall suppose that the characteristic system admits a decomposition map $\pi : M \to B$, i.e., π is a maximal rank, onto mapping with $\pi_*^{-1}(0) = C(I, p)$ for each $p \in M$.

The main theorem describing the geometric properties of the Cauchy characteristic can now be stated as follows:

THEOREM 4.1. *There is a differential ideal of forms, I', on B such that I is the smallest differential ideal on M containing $\pi^*(I')$. In particular, if N is an integral submanifold of I, the set $\pi^{-1}(\pi(N))$ is an integral submanifold (providing it is a submanifold, of course). The maximal integral submanifolds must then contain the fibers of π.*

We shall refer to Cartan's book [6] for the complete proof (although it is not clear whether Cartan's proof takes account of all possible pathology).

To get the idea, let us give the proof in a simple case; suppose, for example, that I is the ideal generated by 1-forms ω_1, ω_2, and a 2-form ω_3. Then, we have relations of the form:

$$d\omega_1 = a\omega_3; \qquad d\omega_2 = b\omega_3.$$

A vector field X that is Cauchy characteristic satisfies

$$\omega_1(X) = 0 = \omega_2(X)$$
$$X \lrcorner\, \omega_3 = c\omega_1 + d\omega_2.$$

Notice that it suffices to work locally. Choose the coordinate system $(x_1, \ldots, x_n)$ so that $X = \partial/\partial x_1$. Then,

$$\omega_1 = \sum_{j=2}^{n} a_j\, dx_j; \qquad \omega_2 = \sum_{j=2}^{n} b_j\, dx_j.$$

$$\begin{aligned} X(\omega_1) &= X \lrcorner\, d\omega_1 = aX \lrcorner\, \omega_3 \\ &= ac\omega_1 + ad\omega_2. \\ X(\omega_2) &= X \lrcorner\, d\omega_2 = bc\omega_1 + bd\omega_2. \end{aligned}$$

Let us look for a combination $\omega = f\omega_1 + g\omega_2$ such that $X(\omega) = 0$. Then,

$$\frac{\partial f}{\partial x_1}\omega_1 + \frac{\partial g}{\partial x_1}\omega_2 = -f(ac\omega_1 + ad\omega_2) - g(bc\omega_1 + bd\omega_2),$$

i.e., f and g satisfy a very simple system of linear ordinary differential equations. Thus, we can find forms ω_1', ω_2' which are a linear combination of ω_1 and ω_2, and which satisfy $X(\omega_1') = 0 = X(\omega_2')$, so that ω_1', ω_2' are forms depending only on the variables $x_2, \ldots, x_n$. Similarly, we can change ω_3 to depend only on these variables. Continue by induction to change the basis of I to depend only on the functions on M defined by pulling up coordinate systems from B via π. q.e.d.

Having described the main geometric properties of the Cauchy characteristics, we now briefly indicate an algorithm which can be used in favorable cases to build up integral manifolds. We focus attention on the *Cauchy problem*: Given an integral manifold N, does there exist an integral manifold of one greater dimension containing N?

Case 1. $\dim C(I, p) =$ constant for all $p \in M$, and $C(I, x) \not\subset N_p$ for all $p \in N$.

Case 2. $\dim C(I, p) =$ constant for all $p \in M$, and $C(I, p) \subset N_p$ for all $p \in N$.

Case 3. The remaining possibilities.

Case 3 we shall ignore, since it only contains the various exceptional and/or singular possibilities. In case 1, we have, by Theorem 4.1, an immediate solution of the Cauchy problem by choosing an $X \in C(I)$ such that $X(x) \notin N_x$ for all $x \in N$ and passing the integral curves of X through each point of N. In case 2, proceed as follows: Consider integrals of $C(I)$, i.e., functions f such that $X(f) = 0$ for all $X \in C(I)$. At least locally, which is all we are concerned with here, there exist such integrals f which are zero on N but are nonconstant. Pick one such f. Consider the differential ideal of forms I, generated by I and f. N is an integral manifold of I_1, and $C(I) \subset C(I_1)$. *It may be possible to choose f so that N satisfies case* 1 *when I is replaced by I_1. In this case, the Cauchy problem for N is solvable with respect to I_1, hence is also solvable with respect to I.*

Notice finally that this method proceeds by successively solving systems of ordinary differential equations, hence works just as well in nonanalytic situations. It is this remark that accounts for the fact that, for example, systems of first-order partial differential equations for one unknown function can be solved by C^∞ methods.

The classical example of this method is that when I is generated by a single one-form ω, i.e., the classical *problem of Pfaff*. We can indicate briefly how this is treated by Cartan's method (first expounded in [1]). First, we will ignore the singularity questions. Suppose then that N is an integral submanifold of I and that case 2 holds. We can divide out if necessary by the Cauchy characteristic system, hence may suppose without loss of generality that I has no nonzero Cauchy characteristic vectors. Now, this implies that the dimension of N is an odd number. Pick *any* hypersurface of M containing N, and restrict I to it. We are now on a space of even number of dimensions, I has nonzero characteristic vectors, and the induction can be continued.*

The singular solutions would be those integral submanifolds N for which this process led at a certain stage to an ideal of forms whose characteristic vectors had higher dimensions. Suppose, for example, that $\dim M = 2r + 1$, and that the characteristic vectors of I are all zero at the generic points of M, but that N is an integral submanifold of I on which there are nonzero characteristic vectors. Let

$$\Omega = d\omega \wedge \ldots \wedge d\omega \wedge \omega \qquad (r \text{ factors of } d\omega).$$

Then, Ω is a multiple $f\theta$ of a $(2r + 1)$-form θ that is everywhere nonzero on M. Now, f must be zero on N; hence, N is an integral manifold of the ideal I' of forms obtained by *adding* f to I.

5. Some Algebraic Properties of the Set of Integral Elements

Putting Cartan's theory to work on some realistic problems in geometry or the theory of partial differential equations depends on having enough information about the algebraic structure of the set of integral elements at each point. These are "only" problems in linear or multilinear algebra, but they usually turn out to be difficult and nonstandard. Cartan never seems to have made a systematic study of this type of problem, but many such computations are scattered through his work. We have found these facts to be among those from Cartan's work that are the most difficult to understand.

In this section, we gather together typical results of this type, to give the reader some idea of the sort of problem involved. Our only contribution here is to provide a free translation of Cartan's arguments into semimodern linear algebra theory. The main problem of significantly extending and generalizing Cartan's work remains.

Most of our work in this section will be concerned with a single real vector space V of finite dimension. The applications to the case where V is the tangent space to a manifold should be evident, but will not usually be specifically mentioned. Let V^* be the dual space of V, and let $\wedge^p V^*$ be the exterior product of p copies of V^*. An element of $\wedge^p V^*$ is then an

* The basic algebraic fact is the following: If V is an even-dimensional real vector space, with I an ideal of skew-symmetric forms on V generated by a nonzero one-form and a two-form, then I has nonzero characteristic vectors.

alternating, p-multilinear form on V. Let $\wedge V^* = \Sigma_{p \geqslant 0} \wedge^p V^*$. If $\omega \in \wedge^p V^*$, $\omega_1 \in \wedge^q V^*$, $\omega \wedge \omega_1$ is defined as a $(p+q)$-form. V^* then is an algebra under $\wedge$, the exterior algebra of V. (We mainly follow Bourbaki notations [1] for linear and multilinear algebra.)

Let I be an ideal of $\wedge V^*$. A subspace $\gamma \subset V$ is an ***integral element*** of I if each $\omega \in I$ is identically zero when restricted to γ. Let $G^p(V)$ be the set of p-dimensional subspaces of V, and let $G^p(I)$ be the subset consisting of p-dimensional integral elements. If $\gamma \in G^p(I)$, let

$$i(\gamma) = \{v \in V : v \lrcorner \omega = 0 \text{ on } \gamma \text{ for all } \omega \in I\}.$$

Let $r(\gamma) = \dim i(\gamma) - \dim \gamma - 1 =$ the dimension of the set of $(p+1)$-dimensional integral elements containing γ. $\gamma_0 \in G^p(I)$ is ***regular*** if $G^p(I)$ is a submanifold of $G^p(V)$ in the neighborhood of γ_0, i.e., if γ_0 is a regular point in the sense of algebraic geometry of the subvariety $G^p(I)$ of $G^p(V)$, and if $r(\gamma_0) = \min_{\gamma \in G^p(I)} r(\gamma)$. Let r_{p+1} denote this minimum value. Let g be the greatest integer such that each integral element of dimension $<g$ is contained in at least one of dimension g. g is called the ***genus*** of I.*

Let $C(I) = \{v \in V : v \lrcorner \omega \in I \text{ for all } \omega \in I\}$. $C(I)$ is a linear subspace of V, the space of Cauchy characteristic vectors of I. Let $V' = V/C(I)$ and let $\pi : V \to V'$ be the projection. The characteristic property of $C(I)$ is that there is an ideal $I' \subset \wedge V'^*$ such that I is the ideal of $\wedge V^*$ generated by $\pi^*(I')$, ($\pi^* : \wedge V'^* \to \wedge V^*$ is the dual linear map to π) and such that $C(I') = (0)$. The genus g' of I' is called the ***reduced genus*** of I, and $g = g' + \dim C(I)$.

For example, if I is generated by a single 2-form ω, g' is equal to the greatest integer h such that $\omega^h \neq 0$ ($\omega^h = \omega \wedge \ldots \wedge \omega$, the exterior product of h copies of ω), $\dim V = 2h$; hence:

$$g = g' + (\dim V - \dim V') = h + \dim V - 2h = \dim V - h.$$

(The proof of this is easily obtained by reducing ω to a canonical form.)

We turn now to some facts that follow easily from the above definitions.

(5.1) *If γ and γ' are integral elements with $\gamma \subset \gamma'$, then $i(\gamma') \leqslant i(\gamma)$. Let $s(\gamma \cdot \gamma') = s(\gamma, \gamma')$. Then, $s(\gamma, \gamma') = r(\gamma) - r(\gamma') - \dim(\gamma'/\gamma)$.*

The proof is obvious from the definitions.

* It would be possible to consider, more generally, a fixed subspace $T \supset V$, only consider those integral elements that are transversal to T, and thus consider the genus with respect to T. For simplicity, we will leave out this refinement.

If γ is an integral element, $\dim \gamma < p$, let $G^p(I, \gamma)$ be the set of $\gamma' \in G^p(I)$ such that $\gamma \subset \gamma'$, i.e., $G^p(I, \gamma)$ is the set of p-dimensional integral elements of I containing γ.

(5.2) *If the pair* $(\gamma_{p-1}^0, \gamma_q{}^0)$ *of integral elements is nonsingular, then* $G^q(I, \gamma_{p-1}^0)$ *is a submanifold of* $G^q(I)$ *in a neighborhood of* $\gamma_q{}^0$. *If* $\gamma_{p-1}^p \subset \gamma_p{}^0 \subset \gamma_q{}^0$ *is a nonsingular triple, then*

$$\dim G^q(I, \gamma_{p-1}^0) = r_p + p - q + \dim G^q(I, \gamma_p{}^0).*$$

Proof. We proceed by induction on $q - p$ to prove that $G^q(I, \gamma_{p-1}^0)$ is a manifold. We know that it is true for $q = p$. (For then, $G^p(I, \gamma_{p-1}^0)$ is isomorphic to the projective space of all lines in $i(\gamma_{p-1})/\gamma_{p-1}$. Let δ be a subspace of $i(\gamma_{p-1}^0)$ of dimension $r_p + 1 + p - q$ such that:

(a) δ and γ_q are in general position.†

(b) $\delta \cap \gamma_{p-1} = (0)$.

(c) $\gamma_p{}^0 = \gamma_{p-1}^0 \oplus \gamma_0 \cap \delta$.

(It is readily verified that such a subspace exists.)

Now, if γ_q is near $\gamma_q{}^0$ and contained in $i(\gamma_{p-1}^0)$,

$$\begin{aligned}\dim(\gamma_q \cap \delta) &= \dim \gamma_q + \dim \delta - \dim i(\gamma_{p-1}^0)\\ &= q + r_p + 1 + p - q - (p - 1 + r_p + 1) = 1.\end{aligned}$$

Then, each such γ_q determines an element $G^p(I, \gamma_{p-1}^0)$, namely $\gamma_{p-1}^0 + (\gamma_q \cap \delta)$, that it contains. $\gamma_q{}^0$ determines $\gamma_p{}^0$, by condition (c). By inductive hypothesis, for each γ_p near $\gamma_p{}^0$, $G^q(I, \gamma_p)$ is a manifold. Thus, $G^q(I, \gamma_{p-1}^0)$ is, near $\gamma_q{}^0$, locally diffeomorphic with

$$G^p(I, \gamma_{p-1}^0) \times G^q(I, \gamma_p{}^0). \quad \text{q.e.d.}$$

(5.3) *If* $\gamma_{p-1}^0 \subset \gamma_{p+2}^0$ *is a nonsingular pair, of integral elements and if* I *is generated by its elements of degree* $\leqslant 2$, *then* $\dim G^{p+2}(I, \gamma_{p-1}) \geqslant 3r_{p+1} - 3$ *in a neighborhood of* γ_{p+2}^0.

* We mean the dimension $G^q(I, \gamma_{0-1}^p)$ in the neighborhood of $\gamma_q{}^0$. A similar remark applies elsewhere in this chapter when we speak of the dimension of real algebraic varieties that may have singular points.

† Recall that two subspaces T_1 and T_2 of a vector space T are said to be in general position if $\dim(T_1 \cap T_2) = \dim T_1 + \dim T_2 + \dim T$. (If the right-hand side is negative, the condition requires that $T_1 \cap T_2 = (0)$.) Then, two subspaces T_1' and T_2' that are, respectively, close to T_1 and T_2 are also in general position.

Proof. First, choose subspaces δ_1, δ_2, $\delta_3 \subset i(\gamma^0_{p-1})$ such that:

(a) $\delta_i \cap \gamma^0_{p-1} = (0)$, $\quad i = 1, 2, 3.$

(b) $\dim \delta_i \cap \gamma^0_{p+2} = 1$, and further δ_i and γ^0_{p+2} are in general position, $i = 1, 2, 3$. This requires that $\dim \delta_i = r_p$.

(c) $\dim \delta_i \cap \delta_j = r_p - 2, i \neq j, j = 1, 2, 3, \dim \delta_1 \cap \delta_2 \cap \delta_3 = r_p - 3$. (This is just a slightly abstract way of saying that γ^0_{p+2} can be obtained from γ^0_{p-1} by adding three linearly independent elements from $i(\gamma^0_{p-1})$.) Let $P(\delta_i)$ be the projective space of lines in δ_i, $i = 1, 2, 3$. There is then one-to-one mapping from a neighborhood U of γ^0_{p+2} in

$$G^{p+2}(I, \gamma_{p-1}) : U \to P(\delta_1) \times P(\delta_2) \times P(\delta_3).$$

Let A_1 be the set of $(l_1, l_2, l_3) \in P(\delta_1) \times P(\delta_2) \times P(\delta_3)$ such that:

(a) l_1 and l_3 are arbitrary.

(b) l_1 and l_2 are in involution, i.e., $l_2 \in i(\gamma^0_{p-1} + l_1)$, $l_2 \notin \gamma^0_{p-1} + l_1$.

Similarly, define A_2 and A_3 by permuting 1, 2, and 3. Now,

$$\dim i(\gamma^0_{p+1} + l_1) \geqslant p + r_{p+1} + 1, \qquad \text{by the definition of } r_{p+1}.$$

For fixed l_1 and l_3, the set of l_2 satisfying (b) is of dimension not less than $r_{p+1} + 1 + (r_p - 1) = r_{p+1} - 1$. Thus, we see that the dimension of A_1 is no less than:

$$2(r_p - 2) + (r_{p+1} - 1).$$

The same inequality holds for the dimension of A_2 and A_3. Note now that:

(a) The image of U in $P(\delta_1) \times P(\delta_2) \times P(\delta_3)$ covers an open set of the intersection $A_1 \cap A_2 \cap A_3$.

(This is where we use the fact that I is generated by its elements of degree $\leqslant 2$, since we must suppose that a subspace of $i(\gamma^0_{p-1})$ is an integral element if all its two-dimensional subspaces are integral elements.)

(b) $\dim(A_1 \cap A_2 \cap A_3)$

$$\geqslant (\dim A_1 + \dim A_2 + \dim A_3 - 2(\dim(P(\delta_1) \times P(\delta_2) \times P(\delta_3))$$

$$\geqslant 3(2r_p - 2) + r_{p+1} - 1) - 6(r_p - 2) = 3(r_{p+1} - 1).$$

This finishes the proof of (4.4), since $\dim U$ is equal to

$$\dim(A_1 \cap A_2 \cap A_3).$$

Remark. We have followed Cartan's proof of (5.3) [2, pp. 266–268]. There might be objections as to the rigor involved in estimating the dimension of intersections of algebraic varieties, which may have singularities, as if they were linear subspaces of a vector space. However, the technique is valid in this case; the corresponding inequalities do hold for the dimensions of the tangent spaces to the varieties, the dimension of U is equal to the dimension of its tangent space, and it is easily seen that the mapping $U \to P(\delta_1) \times P(\delta_2) \times P(\delta_3)$ is an immersion.

From now on, until further notice, we will assume that I is generated by elements of degree $\leqslant 2$.

(5.4) *If* $s_p = r_p - r_{p+1} - 1$, *then* $s_p \geqslant s_{p+1}$.

Proof. From (5.2), we have that:

$$\begin{aligned}\dim G^{p+2}(I, \gamma_{p-1}^0) &= r_p - 2 + \dim G^{p+2}(I, \gamma_p{}^0)\\ &= r_p - 2 + (r_{p+1} - 1) + \dim G^{p+2}(I, \gamma_{p+1}^0)\\ &= r_p - 3 + r_{p+1} + r_{p+2}.\end{aligned}$$

Applying (5.3), we see that:

$$r_p + r_{p+1} + r_{p+2} - 3 \geqslant 3r_{p+1} - 3,$$

hence,

$$s_p = r_p - r_{p+1} - 1 \geqslant r_{p+1} - r_{p-2} - 1 = s_{p+1}. \qquad \text{q.e.d.}$$

(5.5) *If* $s_p = 0$, *then* $0 = s_{p+1} = \dots = s_{n-1}$ *(*n *= genus of* I*)*. The proof follows from the fact that all s_p are $\geqslant 0$ and from the inequalities (5.5).

(5.6) *If* n *is the genus of* I, $s_{n-1} \geqslant r_n$. The proof of (5.7) is similar to that of (5.5), and is left to the reader (or, see Cartan [2]).

(5.7) *If* $s_p = 0$, *then* $n = p + r_p = \dim i(\gamma_{p-1}^0)$, *for each nonsingular* $\gamma_{p-1}^0 \in G^{p-1}(I)$.

$i(\gamma_{p-1}^0)$ *is then the unique n-dimensional integral element containing* γ_{p-1}^0.

Proof. We have i from the fact that $s_p = 0 = s_{p+1} = \dots = s_{n-1}$.

$$r_p = r_{p+1} + 1, \qquad r_{p+1} = r_{p+2} + 1, \dots, r_{n-1} = r_n + 1.$$

Let $A(V)$ be the group of all linear transformations of V into itself. $T \in A(V)$ defines a dual map T^* on p-forms of V. Let $A(I)$ be the subgroup of $A(I)$ composed for the T such that $T^*(I) \subset I$. If $T \in A(I)$, clearly T maps an integral element (resp. integral chain) into an integral element (resp. integral chain), and preserves the notion of regularity and nonsingularity. Also, if $C(I) = \{v \in V : v \lrcorner I \subset I\}$, then $A(I)(C(I) \subset C(I)$.

Much of Cartan's work on the algebraic structure of the integral elements and characteristics of ideals of forms will probably be ultimately seen as contained in various general statements about the action of this group on the various spaces of integral elements. For example, we quote the simplest theorem of this type. Its proof will not be given here, although it can be done quite easily using standard techniques of linear algebra.

THEOREM 5.1. *Let I be an ideal of forms on V generated by a single two-form ω. Let $c = \dim C(I)$. Then, $n - c$ is equal to the rank of the form ω in the sense of linear algebra, and is even. The genus of I is*

$$c + \frac{(n-c)}{2} = \frac{c+n}{2}.$$

$A(I)$ acts transitively on the set of all maximal integral elements of I. All integral elements are regular.

Finally, as a last example of the type of linear algebra involved in Cartan's work (and we have chosen the easiest examples we could find to present here) we present a theorem from Cartan [4], having applications to the theory of partial differential equations, describing a general relation between the characteristic system of a system of quadratic exterior equations and the characteristic system of each of the equations.

Suppose that V is a real vector space of dimension m. Suppose ω_α, $1 \leqslant \alpha \leqslant r$, are linearly independent, skew-symmetric bilinear forms on V. Let

$$C = \{v \in V : v \lrcorner \omega_\alpha = 0 \quad \text{for} \quad 1 \leqslant \alpha \leqslant r\}.$$

For each r-tuple $\lambda = (\lambda_1, ..., \lambda_r)$, let

$$C(\lambda) = \{v \in V : v \lrcorner (\lambda_\alpha \omega_\alpha) = 0\}.$$

Then, $C \subset C(\lambda)$ for each λ.

Suppose $m - \rho = \dim C$, $m - 2n = \min_\lambda(\dim C(\lambda))$. Clearly, if $\dim C(\lambda^0) = m - 2n$, then $\dim C(\lambda) = m - 2n$ for all λ in a neighborhood of λ^0. Let

$$V_1 = \{v \in V : \omega_\alpha(v, C(\lambda)) = 0 \quad \text{for all } \lambda, \text{ all } \alpha\}.$$

THEOREM 5.2. *Suppose that $\rho > 2n$, i.e., there are fewer characteristic vectors of the system of 2-forms ω_α than of each member of the pencil $\lambda_1\omega_1 + ... + \lambda_r\omega_r$. Then $M - n \leqslant \dim V < m$, and every subspace of V of dimension $m - n$ which annuls the system $\omega_\alpha = 0$ is contained in V_1.*

Corollary. *There is a linear subspace W of V with the following properties*:

(a) *W contains C, the set of characteristic vectors of $\omega_1, \ldots, \omega_r$.*

(b) $m - n \leqslant \dim W$.

(c) *Suppose that $\bar{\omega}_\alpha$ denotes the forms ω_α restricted to W, and that $\bar{C}$, $\overline{C(\lambda)}$, $\bar{\rho}$, $\bar{n}$ are the corresponding objects for W. Then, $\bar{C} = \overline{C(\lambda)}$, and* $\dim \bar{C} = 2m - 2n - \dim W$.

(d) *W contains every subspace of V of dimension $m - n$ on which the ω_α vanish identically.*

(e) *W is uniquely determined by these properties.*

Proof of the theorem. First, we show that $V_1 \neq V$. Otherwise,

$$\omega_\alpha(C(\lambda), V) = 0 \qquad \text{for all} \quad \lambda.$$

Choose $v^0 \in C(\lambda^0) - C$. We have $\omega_\alpha(v^0, V) = 0$, i.e., $v^0 \lrcorner\, \omega_\alpha = 0$, hence $v^0 \in C$, contradiction.

Now, suppose that $S \subset V$ is a subspace of dimension $m - n$ such that all the ω_α are zero when restricted to S. Let $v_1, \ldots, v_m$ be a basis for V such that $v_1, \ldots, v_{m-n}$ is a basis for S. Let $\omega^0 = \lambda_1{}^0\omega_1 + \ldots + \lambda_r{}^0\omega_r$. The dimension of the set of linear combinations of the 1-forms $v_1 \lrcorner\, \omega_0{}^0, \ldots, v_m \lrcorner\, \omega^0$ is precisely $2n$. Since $v_1 \lrcorner\, \omega^0, \ldots, v_{m-n} \lrcorner\, \omega^0$ vanish on S, the dimension of their linear combinations is at most $\dim v - \dim S = n$. Combining these two facts, we see that the dimension of the $v_1 \lrcorner\, \omega^0, \ldots, v_{m-n} \lrcorner\, \omega^0$ is *precisely* n, $v_{m-n+1} \lrcorner\, \omega^0, \ldots, v_m \lrcorner\, \omega^0$ are linearly independent, and the space of forms spanned by $v_1 \lrcorner\, \omega^0, \ldots, v_{m-n} \lrcorner\, \omega^0$ is linearly independent from the space of forms spanned by $v_{m-n+1} \lrcorner\, \omega^0, \ldots, v_n \lrcorner\, \omega^0$. Thus, suppose that $v = \sum_{i-1}^m a_i v_j \in C(\lambda^0)$. Then,

$$0 = \sum_{i=1}^m \varphi_i v_i \lrcorner\, \omega^0.$$

These remarks force:

$$\varphi_i = 0, \qquad i \geqslant m - n + 1, \qquad \text{i.e.,} \qquad v = \sum_{i=1}^{m-n} \varphi_i v_i.$$

Thus,

$$0 = \omega_\alpha(v, v_1) = \ldots = \omega_\alpha(v, v_{m-n}).$$

Since this is true for every $v \in C(\lambda^0)$, we see that

$$0 = \omega_\alpha(C(\lambda^0), S) = 0.$$

Since λ^0 is arbitrary, we have that $S \subset V_1$. Now, $C \subset V_1$. Thus, if $\omega_1, \ldots, \omega_r$ are restricted to V_1, the process can be iterated unless

$$\dim C = \min_\lambda \dim C(\lambda, v_1),$$

where $C(\lambda, v_1) = \{v \in V_1 : v \lrcorner (\sum_{k=1}^n \lambda_k \omega_k) = 0\}$ when restricted to V_1. Eventually, the process ends with a subspace we call W. This proves the corollary.

6. Pfaffian and Vector Field Systems

As we mentioned earlier, the most important exterior differential systems for the geometric application are the Pfaffian systems, i.e., systems defined by differential ideals of forms generated by 0- and 1-forms. As Vessiot pointed out [1]*, many of the ideas of the Cartan theory are more natural (from an algebraic point of view, at least) when expressed in terms of the dual notion of vector field system. In this section, we will describe how some of the invariants can be described in these terms, and present, as a sample, Theorem 6.4, perhaps the simplest nontrivial result concerning the invariants. Many similar but harder facts, with interesting applications to the theory of partial differential equations, can be found in Cartan [3].

In this section, H will be a fixed $C(M)$-submodule of $V(M)$, the set of vector fields on a manifold M, such that $p \to H_p$ is a nonsingular vector field system on M. Let I be the differential ideal of forms generated by the one-forms ω that annihilate H, i.e., such that $\omega(X) = 0$ for all $X \in H$.

Lemma 6.1. *Given X, $Y \in H$, $[X, Y] \in H$ if and only if $X(p)$ and $Y(p)$ are in involution with respect to I for all $p \in M$.*

Proof. If ω is a 1-form annihilating H,

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = -\omega([X, Y]).$$

* As a matter of fact, Vessiot developed a different, independent theory for Pfaffian systems, emphasizing the existence of n-parameter families of integral manifolds rather than individual integral manifolds as Cartan did. Vessiot's very interesting work seems to be completely forgotten now.

Now, (Xp) and $Y(p)$ in involution for all p means precisely that $d\omega(X(p), Y(p)) = 0$. q.e.d.

Lemma 6.2. *Given $X \in H$, $[X, H] \subset H$ if and only if X is a Cauchy characteristic vector field for H.*

Proof. Since $X \in H$ already, the condition that X be Cauchy characteristic is that $X \lrcorner d\omega \in I$ for all 1-forms ω that annihilate H. But,

$$X \lrcorner d\omega(Y) = d\omega(X, Y) = -\omega([X, Y]) \qquad \text{for all} \quad Y \in H. \qquad \text{q.e.d.}$$

Lemma 6.3. *Let L be a real subspace of H. Then, $[L, L] \subset H$ if and only if L_p is an integral element for I, for all $p \in M$.*

This follows from Lemma 6.1.

These facts show how the fundamental notions of Section 3 can be interpreted in terms of the Lie algebra structure of $V(M)$ rather than in terms of the Grassman structure of I. Many facts become more intuitive when expressed in this dual way.

For $X \in H$, let $D^1(H, X) = [X, H] + H$, $D^2(H, X) = [X, [X, H]] + [X, H] + H$, etc. Let $D^1(H) = [H, H] + H$, $D^2(H) = D^1(D^1(H))$, etc. $D^1(H)$, $D^2(H)$, ... are the *first, second, ... derived systems of* H. $D(H) = \bigcup_j D^j(H)$, the *derived system of* H. Then, $[D(H), D(H)] \subset D(H)$. $D^1(H, X)$, $D^2(H, X)$, etc., are the *first, second, ... partial derivations of* H *by* X. If $p \in M$, $v \in H_p$ defines $D^1(H, v)$, $D^2(H, v)$, ... as follows: Choose an $X \in H$ with $X(x) = v$. $D^1(H, v) = D^1(H, X)_p$, $D^2(H, v) = D^2(H, X)_p$, etc. This definition is legitimate because it does not depend on the extension X of v that is chosen. Further, for $v \in H_p$

$$\dim H = \dim(D^1(H, v)/H_p) + \dim i(v),$$

hence, the subspace of M_p spanned by v is a regular integral element, if and only if:

$$\dim D^1(H, v) = \max_{\substack{u \in H_p \\ p \in M}} D^1(H, u).$$

If v is a regular integral element,

$$s_1(p) = \dim(H_x/i(v)) = \dim(D^1(H, v)/H_x).$$

We shall assume that $s_1(p)$ is constant for $p \in M$, and denote it by s_1. We shall say that $X \in H$ is *regular* if $X(x)$ is a regular integral element for all $x \in M$. Let dim $D^1(H, X)$ be the constant value of dim $D^1(H, X(x))$:

$$s_1 = \dim D^1(H, X) - \dim H.$$

s_1 is called the *character* of the Pfaffian system H (Cartan [3], von Weber [1]).

THEOREM 6.4. (*von Weber* [1] *and Cartan* [3]). *Suppose that* $s_1(p) = 1$ *for all* $p \in M$. *Let* $X \in H$ *be a regular element of* H. *Then* $D^1(H, X) = D^1(H)$.

Suppose further that

$$\dim H - \dim C(H)_p \,^*$$

* $C(H)$ denotes the set of Cauchy characteristic vector fields in H, i.e.,

$$C(H) = \{X \in H : [X, H] \subset H\}.$$

is constant and greater than 2 *for all* $p \in M$. *Then*, $[D^1(H), D^1(H)] \subset D^1(H)$, *i.e.*, $D^1(H) = D(H)$. *Thus, if* N *is a leaf of the vector field system on* M *defined by* $D(H)$, H *is tangent to* N *and is defined on* N *by a single Pfaffian equation.*

Proof. Suppose first that $X \in H$ is regular, i.e., $[X, H]_x \not\subset H_x$ for all $x \in M$. Since it suffices to prove both of these results locally, we can suppose that $Y \in H$ is such that $[X, Y_0] = Z$ satisfies $Z(x) \notin H_x$ for all $p \in M$. The hypothesis $s_1(p) = 1$ for all $p \in M$ then guarantees that

$$D^1(H, Y_0) = D^1(H, X).$$

To prove the first part, we must show that $D^1(H, Y) = D^1(H, X)$ for all $Y \in H$. Suppose now that we try to find a $Y_1 \in H$ not linearly dependent on X and Y_0 such that $[X, Y_1] \notin H$. For then, we would again have: $D^1(H, Y_1) = D^1(H, X)$. Suppose however that such a Y_1 does not exist. Then, unless X_0 and Y_0 span H, at which point we would be finished, there is a Y' not dependent on X and Y_0 such that, for at least one $p_0 \in M$,

$$[X, Y'](p_0) \in H_{p_0}.$$

Then, $[X, Y' + Y_0] \notin H$, but $Y' + Y_0$ is not dependent on X and Y_0, contradiction. Continuing in this way, we see that there are elements $Y_0, Y_1, \ldots, Y_{s-2}$ which, together with X form a basis for H, such that $D^1(H, Y_j) = D^1(H, X)$ for $0 \leqslant j \leqslant s - 2$. Then, $D^1(H, X) = D^1(H)$.

To prove the second part, suppose now that we choose $Z \in V(M)$ such that H and Z span $D^1(H)$. For $X, Y \in H$, $[X, Y]$ can be written as the sum of an element of H and $\alpha(X, Y)Z$, where $\alpha(X, Y)$ is a real-valued function. Then, $\alpha(X, Y) = -\alpha(Y, X)$, and α passes to the quotient to define a skew-symmetric bilinear form α_p on each H_p:

$$\alpha_p(X(x),\ Y(x)) = \alpha(X, Y)(x).$$

By hypothesis, this form has constant rank as p varies over M. Thus, this form can be brought to canonical form in a smooth way over M. We want to prove that $[D^1(H), D^1(H)] \subset D^1(H)$. But, $[H, H] \subset D^1(H)$. Let $X, Y \in H$ be such that $[X, Y] = Z$, and $Z \notin H$. The assumption that rank $\alpha_p > 2$ implies that there exist at least one other pair $X_1, Y_1 \in H$ such that:

(a) $a(X_1, Y_1) = 1$.

(b) $0 = \alpha(X_1, X) = \alpha(X_1, Y) = \alpha(Y_1, X) = \alpha(Y_1, Y)$.

Then, $[X_1, Y_1] = Z + W$, with $W \in H$, $[X_1, X] \in H$, $(X_1, Y) \in H$, $[Y_1, X] \in H$, and $[Y_1, Y] \in H$. Using these relations, we see that $[X_1, Z] \in [H, H] \subset D^1(H)$. Reversing the role played by X_1 and X, we see that $[X, Z] \in D^1(H)$. Now, given $X \in H$, a Y satisfying these conditions can be chosen unless $X \in C(H)$. But, if $X \in C(H)$, $[X, [H, H]] \subset [H, H]$ by the Jacobi identity. We have then proved that $[D^1(H), D^1(H)] \subset D^1(H)$, as required.

7. A Function-Space Theoretic View of the Singular Solutions

We will discuss the function-space analogue of the following simple geometric fact: If H is a finite-dimensional manifold, if A is a submanifold, and if X is a vector field on H, X is *formally tangent to A* if the value of X belongs to the tangent space of A at each point of A. It then follows that X is actually tangent to A in the sense that the integral curves of X starting at points of A lie completely in A, but the proof uses tools that do not directly generalize to the infinite-dimensional situation. If A is merely a subset of H obtained by setting a number of functions $\{f_i, 1 \leqslant i \leqslant N\}$ on H equal to zero, formal tangency of X to A would mean that $X(f_i) = 0$ on A for $1 \leqslant i \leqslant N$. At the nonsingular points, i.e., at points at which a maximal number of the df_i are linearly independent, formal tangency implies tangency, but the situation is more complicated at the singular points.

Let N and M be manifolds and let $E(N, M)$ be the space of C^∞ maps $\varphi : N \to M$. We will replace H by $E(N, M)$, A by a subset of $E(N, M)$ consisting of all maps satisfying a system of partial differential equations. The notion of "vector field" and "tangent space" of $E(N, M)$ will be formulated below. We will see that the main analytical point in the proof of Cartan's existence theorem for exterior differential systems is just such a question of when "formal tangency" implies actual tangency.

Let $\varphi \in E(=E(N, M))$ be a map: $N \to M$. Let $T(M) = \bigcup_{x \in M} M_x$ be the tangent bundle to M. A *vector field along* φ, typically denoted by v, is a mapping: $N \to T(M)$ such that $v(x) \in M_{\varphi(x)}$ for all $x \in N$. Let E_φ be the set of such vector fields along φ. Since two such vector fields can be added point-by-point, E_φ has a linear structure. Further, if $g \in C(N)$, $v \in E_\varphi$, gv can be defined as an element of E_φ:

$$(gv)(x) = g(x)v(x) \qquad \text{for all} \quad x \in N.$$

Hence, E_φ is a module over the ring $C(N)$.

E_0 may be considered as the *tangent space* to the "infinite-dimensional manifold" E in the following sense: Suppose that φ_t, $0 \leqslant t \leqslant 1$, is a one-parameter family of maps: $N \to M$ with $\varphi_0 = \varphi$. Thus, φ_t may be thought of as a curve in E beginning at φ. The "tangent vector" to this curve at $t = 0$ is an element of E_φ, the infinitesimal deformation as was defined in Section 2, namely

$$\frac{\partial}{\partial t} \varphi_t \bigg|_{t=0}.$$

Note, in general, that E has the following sort of abstract structure. First, a certain class of mappings of finite-dimensional manifolds into E is given. Whenever P is a manifold, ψ a map $N \times P \to M$, a map $P \to E$ is defined by assigning, for each $p \in P$, the map $x \to \psi(x, p) \in M$ for $x \in N$. Let us call a mapping of this type an *injection*. On the other hand, one can recognize a certain natural class of maps of E into finite-dimensional manifolds. Whenever ψ is a map $M \to P$, x is a point of N, the map $E \to P$ can be defined by assigning $\psi(\varphi(x))$ to each $\varphi \in E$. Let us call maps of this type *projections*. The composition of an injection with a projection is a C^∞ map of finite-dimensional manifolds in the ordinary sense, and the system of injections and projections satisfies certain obvious postulates. We can say that any space having the structure given by these postulates is a *differentiable space*. Such a notion can serve to unify all of the various notions of generalized manifold floating around in the literature, but this will be developed further elsewhere. We only mention it here to indicate to the reader that we are treating these function spaces as typical examples of differentiable spaces, i.e., as spaces sharing with ordinary differentiable manifolds the typical curve-tangent vector-vector field structure.

Now, let $T(E) = \bigcup_{\varphi \in E} E_\varphi$ be the *tangent bundle* of E. A tangent vector field *on* E, denoted by V, is a map $E \to T(E)$ such that $V(\varphi) \in E_\varphi$ for each $\varphi \in E$. We have already implicitly encountered such vector fields in Section 2, at least in the case where E is the space of all immersions of $N \to M$. If $\mathbf{X}$ is a map: $G^p(M) \to T(M)$ such that $\mathbf{X}(\gamma) \in M_y$ for each $\gamma \subset M_y$, to each immersion $\varphi : N \to M$ we can associate the following vector field along φ:

$$x \to \mathbf{X}(\varphi_*(N_x)) \qquad \text{for} \quad x \in N.$$

This construction may be generalized using Ehresmann's theory of jets [1]: Let $J^r(N, M)$ be the space of r-jets of maps $N \to M, r = 0, 1, \ldots$. Recall that each such jet is a triple (x, y, φ) consisting of points $x \in N$, $y \in M$ and a map $\varphi : N \to M$ with $\varphi(x) = y$, two such triples (x, y, φ), (x_1, y_1, φ_1) being identified if $x = x_1, y = y_1$, and if the partial derivatives of order $\leqslant r$ at φ and φ_1 coincide at x with respect to local coordinate systems. (We will often denote a jet by $(x, \varphi(x), \varphi)$, leaving to the reader the verification that the result or definition in question is independent of the chosen representative.) Thus, there are projections $J^r(N, M) \to N$ and $J^r(N, M) \to M$, and any map $\varphi : N \to M$ induces a map $\boldsymbol{\varphi} : N \to J^r(N, M)$, where, for each $x \in N$, $\boldsymbol{\varphi}(x)$ is the jet $(x, \varphi(x), \varphi)$.

Any mapping $\mathbf{X} : J^r(N, M) \to M$ such that $\mathbf{X}(x, \varphi(x), \varphi) \in M_{\varphi(x)}$ for all $x \in N$ defines a tangent vector field on $E(N, M)$. To each map $\varphi : N \to M$ we can assign the following tangent vector field along φ:

$$x \to \mathbf{X}(\boldsymbol{\varphi}(x)) \qquad \text{for} \quad x \in N.$$

The concept of "tangent vector field" on a function space $E(N, M)$ leads to the concept of "integral curve" and "one-parameter group" generated by the tangent vector field. Of course, a "curve" in $E(N, M)$ is a one-parameter family φ_t, $\varphi \leqslant t \leqslant b$, of maps of $N \to M$, i.e., a homotopy. It is said to be an "integral curve" of a vector field V on $E(N, M)$ if $(\partial\varphi_t/\partial t) = V(\varphi_t)$ for $a \leqslant t \leqslant b$, i.e., if the "tangent vector" to the curve at each t is equal to the value of V on φ_t. A one-parameter transformation group (or more generally, semigroup) on $E(N, M)$ is said to be generated by a vector field on $E(N, M)$ if all of its orbits are integral curves of the vector field. Thus, the whole apparatus of the Lie theory of transformation groups makes some sort of formal sense on function spaces (and, in fact, on general differentiable spaces).

In particular, we may consider these ideas for vector fields on $E(N, M)$ induced, as explained above, by maps $J^r(N, M) \to T(M)$. It can be shown (Hermann [3]) that the integral curves for such a vector field are determined locally by Cauchy-Kowalewski systems of partial differential equations. The details of defining a Jacobi bracket and exponential map for this type of vector field on function space have been worked out by H. Johnson [1].

One of the main accomplishments of the theory of jets is that the concept of a "general" system of partial differential equations can be formulated in an extremely simple way. Let P be any subset of $J^r(N, M)$. P defines a system of partial differential equations for maps: $N \to M$; a given map $\varphi : N \to M$ is a *solution* of the system if

$$\boldsymbol{\varphi}(x) \in P \qquad \text{for each} \quad x \in N.$$

Let $E(P)$ be the subset of $E(N, M)$ consisting of these solutions. If P is a subset of $J^r(N, M)$ defined by setting a number of functions on $J^r(N, M)$ equal to zero, and if $\varphi \in E(P)$, the vector fields $v \in E_\varphi$ that arise as tangent vectors

$$v = \left.\frac{\partial \varphi_t}{\partial t}\right|_{t=0}$$

to curves $t \to \varphi_t \in E(P)$ satisfy a system of linear partial differential equations, called the *linear variational equations* of the partial differential equations defined by P. Let $E(P)_\varphi$ denote the set of vector fields satisfying this system of partial differential equations [whether or not the vector field is actually tangent to curve in $E(P)$], thought of as the tangent space to $E(P)$ at φ.

For example, suppose that I is a differential ideal of differential forms on M. Let $P \subset J^1(N, M)$ be the set of jets $(x, \varphi(x), \varphi)$ such that $\varphi^*(I) = 0$ at x. Thus, an element $\varphi \in E(P)$ is a map $N \to M$ such that $\varphi^*(I) = 0$, i.e., φ is an integral map for the system of exterior differential equations $I = 0$. The equations defining $E(P)_0$ can be read off from (3.2), namely:

For $\varphi \in E(P)$, a vector field $\mathbf{v} : N \to T(M)$ along φ is in $E(P)_\varphi$, if and only if,

$$d(\varphi^*(\mathbf{v} \lrcorner w)) + \varphi^*(\mathbf{v} \lrcorner dw) = 0 \qquad \text{for all} \quad w \in I.$$

One reason that exterior differential systems are "interesting" from the viewpoint of the theory of general partial differential equations is that the linear variational equations can be written so simply, in a completely global manner!

Now, suppose that $V : E(N, M) \to T(E(N, M))$ is a vector field on $E(N, M)$. We can say that V is *formally tangent* to $E(P)$ if

$$V(\varphi) \in E(P)_\varphi \qquad \text{for all} \quad \varphi \in E(P),$$

and that V is "really" *tangent* to $E(P)$ at an element $\varphi \in E(P)$ if the integral curve φ_t of V in $E(N, M)$ beginning at φ lies in $E(P)$ for sufficiently small t. One recognizes that the basic theorem in Cartan's theory of exterior differential systems, namely Theorem 3.3, asserts, in case P is defined as explained above by a differential ideal I of differential forms on M, that, under certain conditions on φ and V, formal tangency to $E(P)$ implies tangency. Thus, we see that the true geometric significance of the concept of "singular" or "nonsingular" solution of a system of partial differential equations will ultimately be related to the geometry of function spaces, as the analogous geometric problem for finite-dimensional spaces is determined by "singular point" considerations.

8. Systems of First-Order Partial Differential Equations for One Unknown Function

Cartan's theory is easiest to apply to the theory of systems of first-order partial differential equations for one unknown function. In fact, his paper [1] on this subject is his first work on differential systems, hence, presumably was one of the chief examples he had in his mind when developing the general theory.

Let $i, j, \ldots$ be variables running from 1 to n and adopt the summation convention. Also adopt variables $1 \leqslant a, b, \ldots \leqslant m$. Consider variables (x_i, p_i, z) and functions $f_a(x_i, p_i, z)$. We shall also use a vector notation, denoting the functions by $f_a(x, p, z)$, $x = (x_i)$, $p = (p_i)$, etc. We may consider the partial differential equations

$$f_a\left(x_i, \frac{\partial z}{\partial x_i}, z(x)\right) = 0, \tag{8.1}$$

where the unknown is a single function $z(x_i)$. The graph of such a function, i.e., the mapping

$$(x) \to \left(x,\ p = \frac{\partial z}{\partial x},\ z(x) = z\right)$$

is an integral manifold of the following Pfaffian system on (x, p, z)-space:

$$\begin{aligned} dz - p_i\, dx_i &= 0 \\ f_a &= 0 \end{aligned} \tag{8.2}$$

The forms dx_i are obviously independent on such an integral manifold.

Conversely, an n-dimensional integral manifold on which $dx_i, \ldots, dx_n$ are independent, i.e., which is transversal to $dx_i = 0$, determines, at least locally, a solution of the partial differential equations (8.1). For example, if $x = (x_i(t_1, \ldots, t_n))$, $p = (p_1(t_1, \ldots, t_n))$, $z = z(t_1, \ldots, t_n)$ determines such an integral manifold with $dx_i \wedge \ldots \wedge dx_n \neq 0$, obviously the Jacobian $\det(\partial x_i/\partial X_j)$ is $\neq 0$, hence the mapping $t \to x(t)$ can be inverted. The resulting functions $x \to z(t(x)) = z(x)$ solve (8.1).

Now, closing (8.2) under exterior differentiation leads to the system:

$$\begin{aligned} dz - p_i\, dx_i &= 0 \\ dp_i \wedge dx_i &= 0 \\ df_a &= 0 \\ f_a &= 0 \\ dx_i \wedge \ldots \wedge dx_n &\neq 0. \end{aligned} \tag{8.3}$$

Now we are prepared to prove the main theorem about these systems, a result showing that Cartan's definition of "involution" reduces, for these systems, to the more classical definition.

THEOREM 8.1. *Suppose that the rank of the matrix $\partial f_a/\partial p_i$ is everywhere equal to $m(<n)$. Then, the system* 8.3 *is in involution, i.e., these are integral elements of dimension n on which $dx_i \wedge \ldots \wedge dx_n \neq 0$, if and only if all Poisson brackets $\{f_a, f_b\}$ vanish on the submanifold $f_a = 0$, where*

$$\{f_a, f_b\} = \frac{\partial f_a}{\partial x_i}\frac{\partial f_b}{\partial p_i} - \frac{\partial f_a}{\partial p_i}\frac{\partial f_b}{\partial x_i} + p_i\left(\frac{\partial f_a}{\partial z}\frac{\partial f_b}{\partial p_i} - \frac{\partial f_b}{\partial z}\frac{\partial f_a}{\partial p_i}\right).$$

In other words, there should be relations $\{f_a, f_b\} = h_{abc} f_c$, i.e., the ideal (under ordinary multiplication) of functions generated by the f_a is closed under Poisson bracket. Since Poisson bracket satisfies the Jacobi identity, this makes this ideal into a Lie algebra. Although we have foresworn dealing with prolongation problems in this paper, we cannot resist the temptation of pointing out how Theorem 8.1 leads to a solution of the prolongation problem for this case. Suppose then that the system is not in involution. It may still have solutions, of course, but they will be singular, in Cartan's sense. The prolongation problem, in complete generality, asks whether, possibly adding more variables, a system can be constructed including the given system such that the given solution is a regular solution of the new system. Now, in this case, it is (at least under the usual assumptions of maximal rank) not necessary to add new variables. For it is readily verified that the Poisson bracket $\{f_a, f_b\}$ vanishes on the solution. Thus, new systems can be formed including the old by adjoining the Poisson brackets of the functions defining the systems, until finally* a system is obtained whose set of functions defining it is closed under Poisson bracket.

Now we prove the theorem. We assume first that the system is in involution. From the point of view of the theory of exterior differential systems, (8.3) is interpreted as a system on the submanifold $f_a = 0$ closed under d, containing a one-form $\theta = dz - p_i\, dx_i$ and a 2-form $\omega = d\theta = dx_i \wedge dp_i$. Further, it is easily seen that the condition: rank $(\partial f_a/\partial p_i) = m$ implies that θ is nonzero when restricted to this submanifold. Now, by our ealier work, if c denotes the dimension of the characteristic vectors at a point and g the genus, we have

$$g = \frac{(((2n+1)-m)-1)+c}{2} = \frac{2n-m+c}{2}$$

The condition that the system be in involution is that $g = n$ at every point, i.e., that $c = m$. In particular, c has the same value everywhere. Thus, we conclude that there are m vector fields X_a, which are tangent to the submanifold $f_a = 0$, and, when restricted to the submanifold, are linearly independent and are characteristic vector fields for the system $\theta = \omega = 0$.

Explicitly, this means that we have relations of the form

$$\text{(8.4)} \qquad \text{(a)} \quad X_a \lrcorner\, \omega = A_a\theta + A_{ab}\, df_b$$

$$\text{(b)} \quad X_a \lrcorner\, \theta = g_{ab} f_b$$

$$\text{(c)} \quad X_a(f_b) = g_{abc} f_c$$

* A more precise investigation to see whether the Hilbert basis theorem can be used to prove that this process does come to an end would be desirable.

Lemma 8.2. *The basis* (X_a) *for characteristic vector fields can be normalized so that*

$$X_a \lrcorner\, \omega = X_a(x_i)\, dp_i - X_a(p_i)\, dx_i$$
$$= -\frac{\partial f_a}{\partial z}\theta + df_a.$$

Then, we have

$$X_a = \frac{\partial f_a}{\partial p_i}\frac{\partial}{\partial x_i} + \left(\frac{\partial f_a}{\partial z}p_i - \frac{\partial f_a}{\partial x_i}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial f_a}{\partial p_i} + g_{ab}f_b\right)\frac{\partial}{\partial z}. \tag{8.5}$$

Proof. First, we show, if (A_{ab}) is the matrix of functions defined in (8.4), that $\det(A_{ab}) \neq 0$. Otherwise, there are functions λ_a with $\lambda_a A_{ab} = 0$. If $X = \lambda_a X_0$, we then have

$$X \lrcorner\, \omega = (\lambda_a A_a)\theta.$$
$$X \lrcorner\, \theta = (\lambda_a g_{ab}) f_a.$$
$$X(f_b) = (\lambda_a g_{abc}) f_c.$$

The fact that ω does not contain dz, whereas θ does, forces $(\lambda_a A_a) = 0$, i.e., $X \lrcorner\, \omega = 0$. This forces $X(x_i) = X(y_i) = 0$, which in turn forces $X(z) = \lambda_a g_{ab} f_b$, i.e., X, when restricted to the submanifold $f_a = 0$ is identically zero, contrary to our assumption that the X_a are linearly independent when restricted to $f_a = 0$.

Thus, if necessary, the X_a can be replaced by $(A_{ab}^{-1} X_a)$, where (A_{ab}^{-1}) is the inverse matrix of (A_{ab}), so as to satisfy

$$X_a \lrcorner\, \omega = A_a\theta + df_a.$$

Again, the fact that ω does not contain dz forces:

$$A_a = -\frac{\partial f_a}{\partial z}.$$

Substituting what ω and θ are leads directly to (8.5), hence proves the lemma.

Now we are prepared to finish the proof of Theorem 8.1. Notice that using (8.5) and (8.4c) leads to expressing $\{f_a, f_b\}$ in terms of f_c, hence completes the proof one way. But the steps are perfectly reversible; adopt (8.5) as definition of the X_a (where the g_{ab} are determined in terms of the coefficients expressing $\{f_a, f_b\}$ in terms of f_a), and verify that they are characteristic vector fields, i.e., the system has genus n, hence is in involution in Cartan's sense.

Having proved Theorem 8.1, we see that Cartan's basic existence theorem gives a method of "solving" the differential equations (8.2) that we started with. However, this, locally, involves formal power series expressions, and does not really take advantage of the special nature of the system. Both for the purposes of the classical theory, where "explicit" solutions are desired, and for the modern theory, where more information about the existence of merely differentiable solutions and the existence of global solutions is desired, other special "methods" (really, algorithms) of solution are necessary. Lie has done the most profound work on this problem. Although giving the full details would involve us in a full scale exposition of Lie's work, it is too tempting to resist trying to give the main idea here. Consider another system of partial differential equations defined "on" a space of variables (x_i'), defined analytically by functions $f_a'(x_i', p_i', z')$ of variables (x_i', p_i', z'). A mapping from (x, p, z) space to (x', g', z')-space is called a *contact transformation* if the form $dz_i' - p_i' dx_i' = \theta$ is carried back via the mapping to a scalar multiple of the form $\theta = dz - p_i dx_i$, i.e., if the dual mapping maps the Pfaffian equation $\theta' = 0$ onto the Pfaffian equation $\theta = 0$.

We say that such a contract transformation "maps" the system $f_a(x, \partial z/\partial x_i, z) = 0$ into the system $f_a'(x', \partial z'/\partial x', z') = 0$ if the mapping sends the submanifold $f_a = 0$ into the submanifold $f_a' = 0$. The general idea is to try to find such a mapping so that the system $f_a' = 0$ is some standard system that can be explicitly and easily solved. Very intuitively speaking, the "bigger" the *group* of all contact transformations of (x, p, z)-space into itself mapping the given system into itself the "easier" it is to find such a contact transformation to a simple, standard system. However, from a modern point of view, this group (more precisely, an "infinite Lie pseudogroup") should be the primary object of interest.

9. The Isometric Imbedding Problem

Although Cartan's theory leads to interesting new outlooks on the classical theory of partial equations, its real genius and power only comes into force when more purely geometric problems are considered. One reason for this is that the theory is completely free of local coordinates. Even more remarkable, however, is the fact that setting up a problem according to Cartan's prescription leads naturally and quickly to the most important geometric invariants and concepts associated with the problem. This claim can be most readily documented by Cartan's own treatment of classical surface theory in [6]; to supplement this we will give here a presentation of the problem of isometrically imbedding one Riemannian manifold in another, based on Cartan's work in [5]. We will carry the discussion to the point where proving the isometric imbedding is reduced to a problem in linear algebra. Actually, solving this linear algebra problem is admittedly the most difficult part of the proof, but at the present time there does not seem to be a way to simplify Cartan's solution substantially. Here again, we see the need for further development of the underlying algebraic theory. The situation, historically speaking, is analogous to that in Lie algebra theory before the highly ingenious pioneering work of Lie, Killing, and Cartan had been well understood by the professional algebraists.

Let M be a manifold of dimension m. A *Riemannian metric* is defined on M by a tensor field on M whose value at each point $x \in M$ is a positive definite, symmetric bilinear form $(u, v) \to \langle u, v\rangle$ on the tangent space M_x. (If the inner product form $\langle\ ,\ \rangle$ is only nondegenerate, but not necessarily of constant sign, the geometric structure is called a *pseudo-Riemannian metric*. Many of the formal, local properties of Riemannian metrics carry over to the pseudo-Riemannian situation, but the global theory seems to be of a completely different nature. This difference is very analogous (and, of course, related) to the difference between elliptic and hyperbolic partial differential equations.)

There seem to be two distinct approaches to the problem of defining the local differential geometric invariants of a Riemannian metric. The first works via the notions of affine connection and covariant derivative, more-or-less equivalent to classical tensor analysis except that independence of local coordinate systems is emphasized and made very explicit instead of being hidden under a maze of special conventions. This approach can be most readily found in Helgason's book [1]. The second approach works by constructing a fiber bundle over M, the *orthonormal frame bundle*, and constructing a so-called Cartan connection on this bundle. As the name indicates, this approach is due to Cartan himself, and is the one best suited to applying Cartan's theory of exterior differential systems. (In fact, we may conjecture that one guiding general principle in Cartan's work on differential geometry was a search for a formulation to which the theory of exterior differential systems could be most readily applied.) At the present time, it appears that the affine-connection approach is best suited to computation, whereas the bundle approach is valuable for obtaining geometric insight into new problems. Hence, a sketch of the relation between the two approaches may be in order here.

An *affine connection* on M is defined as an R-bilinear mapping $V(M) \times V(M) \to V(M)$, denoted by $(X, Y) \to \nabla_X Y$, such that

$$\text{(9.1a)} \qquad \nabla_{fX} Y = f \nabla_X Y$$

$$\text{(9.1b)} \qquad \nabla_X(fY) = X(f)Y + f\nabla_X Y \qquad \text{for } f \in C(M), \quad X, Y \in V(M).$$

There is a unique affine connection on M associated with a given Riemannian metric $\langle\ ,\ \rangle$, such that

$$\text{(9.2a)} \qquad X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

$$\text{(9.2b)} \qquad \nabla_X Y - \nabla_Y X = [X, Y], \qquad \text{for all } X, Y, Z \in V(M).$$

$\langle Y, Z\rangle$ is the function on M such that $\langle Y, Z\rangle(x) = \langle Y(x), Z(x)\rangle$ for $x \in M$. (The *inner product* of Y and Y, corresponding to the ordinary dot product of vector fields in classical vector analysis.) Condition (9.2a) may be described by saying that the metric tensor has vanishing covariant derivative with respect to the affine connection. Condition (9.2b) is the condition that the torsion tensor of the affine connection be zero.

We call this affine connection *the* Riemannian connection. (It may also be called the Levi-Civita connection, after its actual discoverer.) We can make several auxiliary definitions. First, if $p \in M$, $v \in M$, $Y \in V(M)$, define $\nabla_v Y$ as an element of M_p as follows:

Choose any $Y \in V(M)$ with $Y(p) = v$, and put

$$\nabla_v Y = \nabla_X Y(p).$$

Condition (9.1a) guarantees that this definition is independent of how v is extended to a vector field on M. Second, define the *curvature tensor* as a $C(M)$-multilinear map $V(M) \times V(M) \times V(M) \to V(M)$, denoted by $(X, Y, Z) \to R(X, Y)(Z)$, as follows:

$$\text{(9.3)} \qquad \nabla_X \nabla_Y(Z) - \nabla_Y \nabla_X(Z) = R(X, Y)(Z).$$

The fact that R so defined is a tensor field, i.e., is $C(M)$-multilinear, is not automatic, but must be verified using (9.1). Knowing this, it is possible to define its *value* at each point $p \in M$, assigning $R(u, v)(w) \in M_p$ to all $u, v, w \in M_p$, as follows:

$R(u, v)(w) = R(X, Y)(Z)(p)$ for any choice of X, Y, $Z \in V(M)$ such that $X(p) = u$, $Y(p) = v$, $Z(p) = w$.

The (orthonormal) *frame bundle* $F(M)$ of the Riemannian metric on M is defined as follows: An element e of $F(M)$ consists of an ordered pair $(p; (v_1, \ldots, v_m))$, where $p \in M$ and $(v_1, \ldots, v_m)$ is an orthonormal basis of the tangent space to M at x, i.e.,

$$\langle v_i, v_j \rangle = \delta_{ij}.$$

(Adopt the range of indices $1 \leqslant i, j, \ldots \leqslant m$ and the summation convention.) It is readily verified that $F(M)$ can be made into a differentiable manifold, and that the projection map $\pi : F(M) \to B$ given by $\pi(b; (v_1, \ldots, v_m)) = b$ is a maximal rank mapping. If U is an open subset of M, a cross-section map $\varphi : U \to F(M)$ evidently determines a basis $(X_1, \ldots, X_n)$ of vector fields in U such that

$$\varphi(x) = (p, X_1(p), \ldots, X_n(p)) \tag{9.4a}$$

$$\langle X_i(p), X_j(p) \rangle = \delta_{ij} \qquad \text{for} \quad p \in U. \tag{9.4b}$$

Conversely, such a set of orthonormal vector fields determines a cross-section map. Cartan would call these a *moving frame* (*répère mobile*) for the Riemannian structure, and, not having the language of fiber bundles at his disposal, he was forced to always state things in terms of local moving frames. (This remark is typical of all of Cartan's work on differential geometry.) Let θ_i be the 1-forms in U dual to the X_i, i.e.,

$$\theta_i(X_j) = \delta_{ij}.$$

Then, the metric in U is given by $\theta_i \circ \theta_i$ ($\circ$ means symmetric product) in the sense that

$$\langle u, v \rangle = \theta_i(u)\theta_i(v) \qquad \text{for} \quad p \in U, \quad u, v \in M_p.$$

It can be readily verified that there is a unique set of everywhere independent 1-forms on $F(M)$, denoted by $w_1, \ldots, w_n$, such that, for every such cross-section map φ,

$$\varphi^*(w_i) = \theta_i.$$

For a basis (X_i) of vector fields satisfying (8.3), define 1-forms θ_{ij} in U as follows:

$$\theta_{ij}(X) = \theta_i(\nabla_X X_j) = \langle X_i, \nabla_X X_j \rangle.$$

The following relations are easily derived by direct computation from (8.1–8.4):

(9.5a) $\theta_{ij} + \theta_{ji} = 0$ (equivalent to 8.2a)

(9.5b) $d\theta_i = \theta_{ij} \wedge \theta_j$ (equivalent to 8.2b)

(9.5c) If $\Omega_{ij} = d\theta_{ij} - \theta_{ik} \wedge \theta_{kj}$, then

$$R(X_k, X_l)(X_i) = \Omega_{ij}(X_k, X_l)X_j \qquad \text{(equivalent to 8.3).}$$

The forms θ_{ij} and Ω_{ij} defined in U are called the *connection forms*, and *curvature forms*, with respect to the Riemannian connection ∇ and the moving frames X_i (or θ_i). When the moving frames are changed, the θ_{ij} satisfy a simple (but nontensorial) transformation law, which we shall not give explicitly. Instead, the θ_{ij} may be described as follows:

There are 1-forms w_{ij} *globally defined* on $F(M)$ such that

(9.6a) $w_{ij} + w_{ji} = 0.$

(9.6b) The forms w_i and w_{jk}, $1 \leqslant i \leqslant n$, $1 \leqslant j < k \leqslant n$, together form a basis for 1-forms on $F(M)$.

(9.6c) $dw_i = w_{ij} \wedge w_j.$

(9.6d) $dw_{ij} = w_{ik} \wedge w_{ki} + R_{ijkl}w_k \wedge w_l$, for some functions R_{ijkl} on $F(M)$.

(9.6e) For any cross section $\varphi : U \to F(M)$ determined by a moving frame (X_i) in U,

$$\varphi^*(w_{ij}) = \theta_{ij}, \qquad \varphi^*(w_i) = \theta_i.$$

These forms (w_i, w_{ij}) globally defined on $F(M)$ determine the Cartan connection on $F(M)$ mentioned earlier, but to describe this in more detail here would take us too far afield. Note from (9.6) how the curvature forms Ω_{ij} and hence, via (9.5c), the curvature tensor, are determined by means of the functions R_{ijkl} on $F(M)$:

$$\varphi^*(R_{ijkl}w_k \wedge w_l) = \Omega_{ij}.$$

(Actually, the R_{ijkl} are the *components* of the tensor field, in the sense of classical tensor analysis, made into *global* functions by some sort of lifting process to a space, namely $F(M)$, sitting over M.)

The work up to now has described the differential geometric invariants of the Riemannian metric on M. We must extend this to study a submanifold N of M, with $\dim N = n$, $n < m$. Now, N itself has a Riemannian metric induced from that on M: If $i : N \to M$ is the map defining N as a submanifold,

$$\langle u, v\rangle = \langle i_*(u), i_*(v)\rangle \qquad \text{for} \quad p \in N, \qquad \text{all} \quad u, v \in N_p.$$

For notational convenience, we will identify N with its image in M under i, and identify N_p with its image $i_*(N_p)$ in $M_{i(p)}$, i.e., in M_p. If we start off with a given metric on N, and if this metric agrees with this induced metric, we say that the given Riemannian manifold N is *isometrically imbedded* in M. For the moment, we will work with such an isometric imbedding. The conditions we will find will then be turned around to set up an exterior differential system whose solution will give an isometric imbedding.

For $p \in N$, let $N_p{}^\perp$ be the subspace of M_p consisting of the vectors that are perpendicular to N_p with respect to the form $\langle\ ,\ \rangle$. We now define the *second fundamental form* of N. (The *first fundamental form*, in classical language, is just the metric tensor $\langle\ ,\ \rangle$ of M restricted to N_p.) Algebraically, it will be an R-linear function assigning to each $u \in N_p{}^\perp$, $p \in N$, a symmetric bilinear form $S_u(\ ,\ )$ on N_p. As definition, for v_1, $v_2 \in N_p$, choose X, $Y \in V(M)$ such that

(a) $\qquad X(p) = v_1, \qquad Y(p) = v_2,$

(b) $\qquad X$ and Y are tangent to N,

and define

$$S_u(v_1, v_2) = \langle u, \nabla_X Y(p)\rangle. \tag{9.7}$$

It is readily verified that this is independent of the extensions chosen. The symmetry of $S_u(\ ,\ )$ follows at once from (9.7), since, using (9.2b),

$$\begin{aligned} S_u(v_1, v_2) &= \langle u, \nabla_Y X + [X, Y](x)\rangle \\ &= \langle u, \nabla_Y X(p)\rangle = S_n(v_2, v_1). \end{aligned}$$

(Use the fact that $[X, Y]$ is also tangent to N.) The curvature tensor $R'(\ ,\ )(\ )$ of the induced metric on N can be determined in terms of

$R(\ ,\ )(\ )$, the curvature tensor of M, and S. In fact, by a straightforward (but tedious) computation using the formulas developed above, we have

$$(9.8) \qquad \langle v_1, R'(v_2, v_3)(v_4)\rangle - \langle v_1, R(v_2, v_3)(v_4)\rangle$$
$$= \sum_{i=n+1}^{m} S_{u_i}(v_3, v_4)S_{u_i}(v_1, v_2) - S_{u_i}(v_2, v_4)S_{u_i}(v_1, v_3),$$
$$\text{for} \quad v_1, v_2, v_3, v_4 \in N_p$$

where $u_{n+1}, \ldots, u_m$ is any *orthonormal* basis of N_x. (This formula, specialized to the case $n = 3$, $m = 2$, gives the famous Theorema Egregium of Gauss.)

Let U be an open subset of M such that $U \cap N$ is nonempty. A moving frame, for the Riemannian structure of M, defined in U as an orthonormal basis (X_i) of vector fields in U, is said to be *adapted* to N, if:

$$(9.9) \qquad X_1, \ldots, X_n \text{ are tangent to } N \cap U.$$

It follows, of course, from the orthonormality of the X_i that the vector fields $X_{n+1}, \ldots, X_m$ are perpendicular to N.

(The moving frames that are adapted to N may be thought of as the moving frames for the geometric structure described by the pair (M, N) of Riemannian manifold and submanifold.) Let $F(N)$ be the orthonormal frame bundle of N with respect to its metric. It, of course, does not depend on the imbedding, but only on the metric on M. Choose the following additional indices and the corresponding summation conventions:

$$1 \leqslant a, b, \ldots \leqslant n; \qquad n + 1 \leqslant u, v, \ldots \leqslant m.$$

Let (w_a', w_{ab}') be the global forms on $F(N)$ defining, as explained above for M, the Cartan connection on $F(N)$.

Now, any orthonormal basis (X_i) of vector fields in U that is adapted to N defines a map

$$\psi : U \cap N \to F(N) \times F(M)$$

as follows:

> For $p \in U \cap N$, the $F(N)$-component of $\psi(p)$ is the orthonormal basis of N_p given by $X_1(p), \ldots, X_n(p)$, while the $F(M)$-component of $\psi(p)$ is the orthonormal basis of M_p given by $X_1(p), \ldots, X_m(p)$.

Now, $\psi(U \cap N)$ is a submanifold of $F(N) \times F(M)$ (since ψ is a cross-section map) of dimension n having, it is readily verified, the following properties:

(9.10a) $w_a - w_a' = w_u = 0$ when restricted to $\psi(U \cap N)$, $1 \leqslant a \leqslant n$, $n - 1 \leqslant u \leqslant m$.

(9.10b) $w_1', \ldots, w_n'$ are everywhere linearly independent when restricted to $\psi(U \cap N)$.

(It is assumed, without any change in notation, that forms on $F(N)$ and $F(M)$ are considered as forms on $F(N) \times F(M)$ by pulling back via the projection map.) This says that $\psi(U \cap N)$ is an integral manifold of the Pfaffian system (9.10a), with w_a' linearly independent on the submanifold. *Conversely*, any n-dimensional submanifold satisfying these conditions defines a local isometric imbedding of N (considered now with its own, independent metric) into M: We would only have to project the submanifold down to N, verify using (9.10b) that the projection map restricted to the submanifold had maximal rank, hence was onto an open subset of N, then verify that the inverse map to this projection, followed by the projection on M, was the desired isometric imbedding.

Now, (9.10a) is not closed under (b). Closing it up can be done using (9.6) (and the corresponding formulas for $F(N)$). The result is easily verified to be:

(9.11a) $$w_a' - w_a = w_u = w_{ab} - w_{ab}' = w_{ua} \wedge w_a.$$

(9.11b) $$w_{ua} \wedge w_{ub} + (R_{abcd}' - R_{abcd}) w_c \wedge w_d = 0.$$

(9.11c) $$w_1' \wedge \ldots \wedge w_n' \neq 0.^*$$

THEOREM 9.1. (Janet [1], Cartan [5]). *If* $\dim M = m = [n(n-1)]/2$ *the exterior differential system* (8.11) *is in involution, i.e., if N and M are real analytic there are regular integral manifolds of dimension n of the Pfaffian system* (8.11) *on which the forms $w_1', \ldots, w_n'$ are everywhere independent. Each such integral manifold defines* (*locally*) *an isometric imbedding of N in M.*

In case M is Euclidean space, this theorem was first proved by Janet, although Cartan states that the discovered it independently. However, Cartan's proof, which we are following, has the great merit of applying

* Recall that this notation means that we are looking for an n-dimensional integral manifold of (8.11a) on which the forms $w_1', \ldots, w_n'$ are linearly independent.

also to pseudo-Riemannian metrics and to the case where M is an arbitrary Riemannian manifold.* A recent proof for the case where M is Euclidean space has been given in the classical language by Friedman [1]. Of course, there are also the well-known results of Nash [1] concerning C^∞ global isometric imbedding, which are, as far as we know, restricted to the case where M is Euclidean and of a considerably higher dimension that $[n(n+1)]/2$.

We will refer to Cartan's original paper [5] for the solution of the algebraic problem involved in showing that there is a plentiful supply of integral elements.

References

N. Bourbaki

1. Algèbre linéaire et multilinéaire, *in* "Éléments de mathématique," Volume II, Chapters 2, 3, Hermann, Paris, 1949.

E. Cartan

1. Sur certaines expressions différentielles et le problème de Pfaff, *Ann. École Norm. Sup.* **16** (1899), 239-332; "Œuvres Complètes," Part II, Vol. 1, 303-396, Gauthier-Villars, Paris, 1958.
2. Sur l'intégration des systèmes d'équations aux différentielles totales, *Ann. École Norm. Sup.* **18** (1901), 241-311; "Œuvres Complètes," Part II, Vol. 1, 411-482, Gauthier-Villars, Paris, 1958.
3. Sur l'intégration de certaines systèmes de Pfaff de caractère deux, *Bull. Soc. Math. France* **29** (1901), 233-302; "Œuvres Complètes," Part II, Vol. 1, 483-554, Gauthier-Villars, Paris, 1958.
4. Sur les transformations de Bäcklund, *Bull. Soc. Math. France* **43** (1915), 6-24; "Œuvres Complètes," Part II, Vol. II, 1175-1194, Gauthier-Villars, Paris, 1958.
5. Sur la possibilité de plonger un espace Riemannien donné dans un espace euclidien, *Ann. Soc. Pol. Math.* **16** (1927), 1-7; "Œuvres Complètes," Part III, Vol. 2, 1094-1098, Gauthier-Villars, Paris, 1958.
6. "Les systèmes différentielles, extérieures et leurs applications géométriques," Hermann, Paris, 1946.

C. Chevalley

1. "Lie Groups I," Princeton Univ. Press, Princeton, New Jersey, 1946.

C. Ehresmann

1. Introduction à la théorie des structures infinitésimales et des pseudogroupes de Lie, "Colloque de géométrie différentielle de Strasbourg," Centre Natl. Rech. Sci., 1953, 97-117.

* Cartan does not seem to have noticed that his proof applies if M is arbitrary, although he does mention that it applies also to the pseudo-Riemannian case.

A. Friedman

1. Local isometric imbedding of Riemannian manifolds with indefinite metrics, *J. Math. Mech.* **10** (1961), 625-649.

A. Haefliger

1. Structures feuilletées et cohomologie à valeurs dans un fasceau de groupoïdes, *Comm. Math. Helv.* **32** (1958), 248-329.

S. Helgason

1. "Differential Geometry and Symmetric spaces," Academic Press, 1962.

R. Hermann

1. The differential geometry of foliations, *Ann. of Math.* (1960), 445-457; Part II, *J. Math. Mech.* (1962), 303-316.
2. Fundamental domains for Riemannian transformation groups, *Proc. Am. Math. Soc.* **13** (1962), 489-494.
3. Formal tangency of vector fields in function spaces. Technical report, Univ. of California, Berkeley, 1962.

M. Janet

1. Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien, *Ann. Soc. Pol. Math.* **5** (1926), 38-73.

H. H. Johnson

1. Bracket and exponential for a new type of vector field, *Proc. Am. Math. Soc.* In press.

E. Kähler

1. "Einführung in die Theorie der Systeme von Differentialgleichungen," Chelsea, New York, 1949.

M. Kuranishi

1. On E. Cartan's prolongation theorem of exterior differential systems, *Am. J. Math.* **79** (1957), 1-47.
2. On the local theory of continuous infinite pseudogroups, I and II, *Nagoya Math. J.* **15** (1959), 225-260; **19** (1961), 55-91.

J. Nash

1. The imbedding problem for Riemannian manifolds, *Ann. Math.* **63** (1956), 20-63.

R. Palais

1. A global formulation of the Lie theory of transformation groups, *Mem. Am. Math. Soc.*, No. 22 (1957), Chapter I.

G. Reeb

1. Sur les espaces fibrés et les variétés feuilletées, *Actualités Sci. Ind.* No. 1183 (1952).

B. Reinhart

1. Foliated manifolds with bundle-like metrics, *Ann. Math.* **69** (1959), 119-132.

E. Vessiot

1. Sur une théorie nouvelle des problèmes généraux d'intégration, *Bull. Soc. Math. France* **52** (1924), 336-395.

E. von Weber

1. Zur Invariantentheorie der Systeme Pfaffs'cher Gleichungen, *Leipzig Ber.* (1898), 207-229.